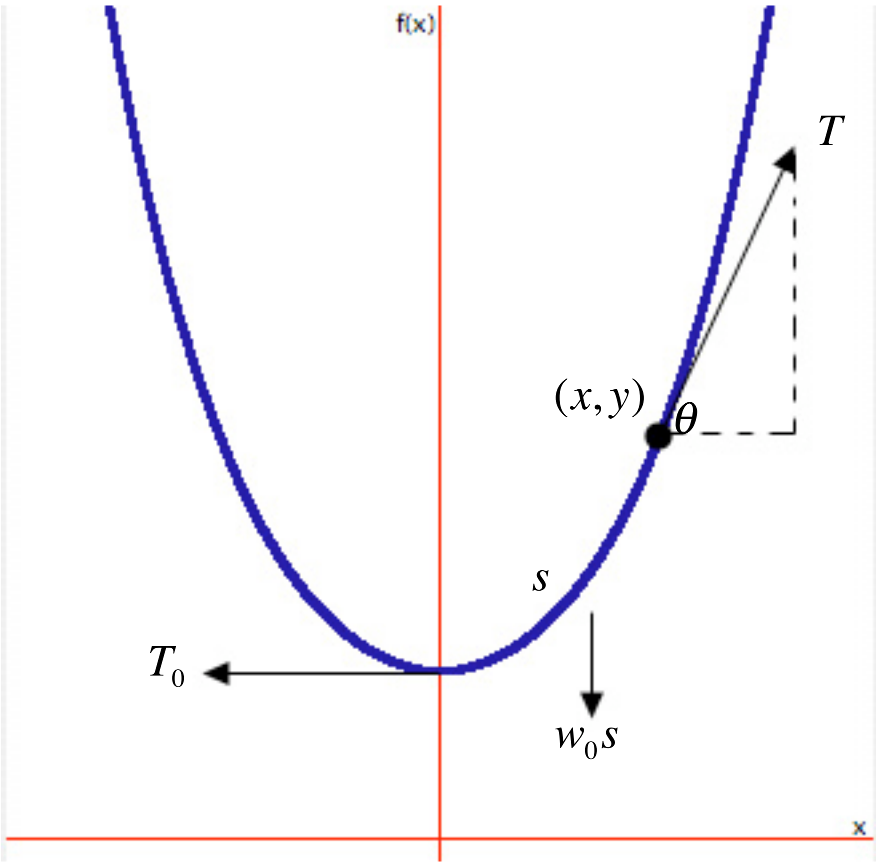
Calculus

The Catenary

A classical problem that involves the application of hyperbolic functions is to determine the exact shape of a *catenary*. Hmmm, what is a *catenary* you might ask. It is a flexible chain of uniform density which is suspended between two points and hangs under its own weight. It comes from the latin word *catena*.

Many mathematicians thought at that time that the shape would follow a x^2 curve. Ha, it isn't that easy.

We set our coordinate system such that the y-axis passes through the lowest point of the chain. Let *s* be the arc length from this point to a variable point (*x, y*), and let w_0 be the linear density of the chian, that is the weight per unit length. Here is where some physics is used. The part of the chain from the lowest point to the variable point (*x, y*) is in static equilibrium under the action of three forces: the tension T_0 at the lowest point directed horizontally left; the variable tension T at (*x, y*) directed in the tangential direction of the curve; and the downward force $w_0 s$ from the weight of the chain. A diagram illustrates these forces.



We'll now do what physics students call, "resolving forces". Since this part of the chain is in static equilibrium, we can equate the forces in the horizontal and vertical direction. This gives us,

$$T\cos\theta = T_0 \text{ and } T\sin\theta = w_0 s$$

We divide the equations to eliminate T and get

$$\tan\theta = \frac{w_0 s}{T_0}$$

Since $\tan\theta$ describes the gradient of the curve at a certain point, we can write

$$\frac{dy}{dx} = as \quad \text{where} \quad a = \frac{w_0}{T_0}$$

replacing the constants with a convenient *a*.

Here is where some differentiation intuition is needed. We want to eliminate the *s* term in the equation but it will be clumsy to substitute the explicit formula for the arc length. How about substituting the differential form *ds* which is much neater. To do this, we need to differentiate w.r.t *x*.

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\frac{\sqrt{dx^2 + dy^2}}{dx}$$
$$\frac{d^2y}{dx^2} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

This is the differential equation of the catenary. Solving this gives us the required shape. We can do that be two successive integrations. First, we need to introduce the auxiliary variable $p = dy/dx$, so we now have

$$\frac{dp}{dx} = a\sqrt{1 + p^2}$$

By separating the variables and integrating we get

$$\int \frac{dp}{\sqrt{1+p^2}} = a\int dx$$

Remember that nice little integral formula we got from our hyperbolic sine. Well, we'll use it now. The equation above follows that form so we immediately write

$$\sinh^{-1} p = ax + c_0$$

We see from the graph that when $x = 0$, where the curve intersects the y-axis, the graph is zero or $p = 0$ as we earlier define $p = dy/dx$. It follows that c_0=0 and so the equation reduces to

$$p = \sinh ax$$

Replacing *p* we then solve the differential equation:

$$\frac{dy}{dx} = \sinh ax$$

$$dy = \sinh ax\, dx$$

$$y = \int \sinh ax\, dx$$

$$y = \frac{1}{a}\cosh ax + c_1$$

We would like to make the equation a little neater so what we'll do is that we'll move the coordinate axis such that the origin is at just the right level of $y = 1/a$ whe $x = 0$. This gives us $c_1 = 0$ and finally the shape of the catenary takes the form

$$y = \frac{1}{a}\cosh ax$$

Who would have thought an innocent chain hanging on its two ends is defined by something as usual as the hyperbolic cosine function. Now you can tell your friends that a catenary takes this form, and not the common x^2 everyone thought of.