Differential Vector Calculus

The Fundamental Theorem of Calculus

All high school calculus courses will most probably start a discussion on the integral calculus with the Riemann sums. While it does work by taking the sum of small little areas below the graph, we get a rather complicated definition of the definite integral as

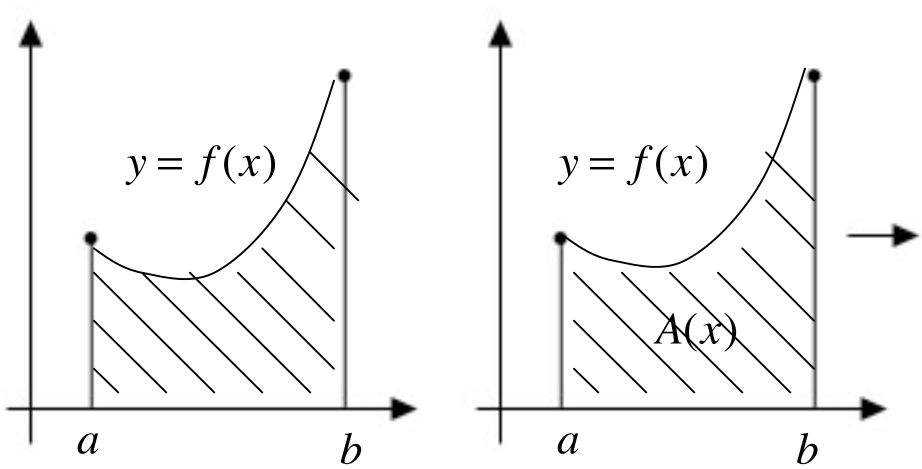
$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

While this method does work, it is impractical and in some case even impossible to use when evaluating complicated integrals such as

$$\int_0^3 \frac{x^5}{\sqrt{1-x^2}}dx \text{ and } \int_0^{3\pi} \sin\left(\frac{x}{x-4x^2}\right)dx$$

Thus begs the question, how do we progress in evaluating these integrals, and in turn finding the area under the graph, bearing in mind that they didn't have calculators in those days. Nonetheless, from the mind of Newton and Leibniz came a new method.

Their method of calculating the definite integral of such functions seems paradoxical in first sight. The approach was to ask a more difficult question: Instead of considering a fixed area like the graph on the left, how do we calculate the variable area, denoted by $A(x)$, when the right side of the area is considered movable.



The objective here is to find $A(x)$ where the area is a function of x. Clearly $A(a)=0$ and $A(b)$ is the fixed area in the graph on the left. We need to find an explicit formula for $A(x)$ and then determine the area by letting $x=b$. This is a two-step process. You ready? Then let's go!

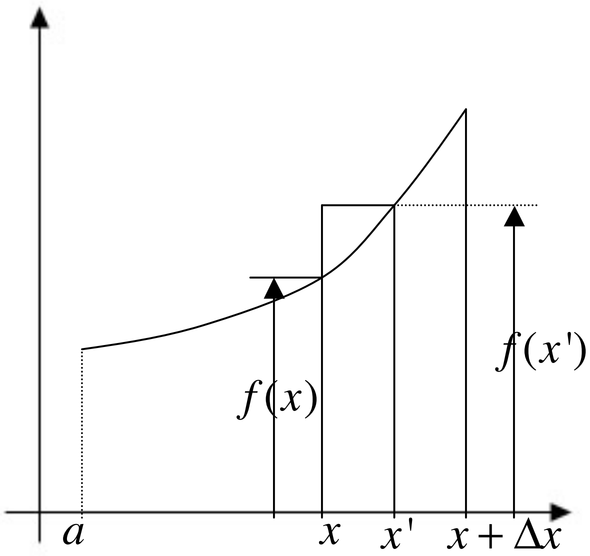
Step 1. We need to establish the fact that

$$\frac{dA}{dx}=f(x)$$

which says that the rate of change of area A with respect to a change in x is equal to the length to the right of the region, or equivalently $f(x)$. Well this may or may not be obvious we shall prove it here. Recall differentiating by first principles,

$$\frac{dA}{dx}=\lim_{\Delta x\to 0}\frac{A(x+\Delta x)-A(x)}{\Delta x}$$

Now $A(x)$ is the area under the graph from a to x and $A(x+\Delta x)$ is the area between a and $x+\Delta x$. And thus the expression $A(x+\Delta x)-A(x)$ is the area between x and $x+\Delta x$. By the mean value theorem, we see that this area is exactly equal to the area of a rectangle with the same base whose height is $f(x')$ where x' lies between x and $x+\Delta x$.



Using the graphical reasoning above, we can write

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x')\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} f(x') = f(x)$$

since $f(x)$ is continuous. For further elaboration, $\Delta x \to 0$ is equivalent to $x + \Delta x \to x$. Since x' is caught between x and $x+\Delta x$, we also have $x' \to x$, and in turn $f(x') \to f(x)$.

Step 2. In the previous step, we have the equation

$$\frac{dA}{dx} = f(x)$$

This allows us to find a formula for the area $A(x)$. Now, $A(x)$ is one of the antiderivatives of $f(x)$. I emphasize ONE of the antiderivatives because we know from differentiating that is that we can differentiate a any constant and get the same result. So now say that $F(x)$ is *any* antiderivative of $f(x)$ and thus

$$A(x) = F(x) + c$$

To determine c, we let $x = a$ knowing that this will give us $A(a) = 0$ and so $c = -F(a)$. Therefore

$$A(x) = F(x) - F(a)$$

Reintroducing the point b as we have shown on the graph, recognize that

$$\int_a^b f(x)dx = A(b) = F(b) - F(a)$$

And so I present to you the Fundamental Theorem of Calculus:

If $f(x)$ is continuous on a closed interval $[a,b]$, and if $F(x)$ is any antiderivative of $f(x)$ so that $(d/dx)F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Out of convenience and convention, we can also write

$$\int f(x)dx = F(x)$$

though I must stress that this actually means from the antiderivative of $f(x)$. We can now reduce the problem of evaluating definite integrals of complicating functions to finding their antiderivatives.