Calculus

Derivatives of Hyperbolic Functions

Now that we got some basic definitions of hyperbolic functions, we shall move swiftly along and look at their derivatives and integrals. It turns out that these functions, which I remind you again have been discouraged for students to learn, turns out to be highly useful to finding certain integrals.

The derivatives are similar to those of trigonometric functions though not entirely the same:

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}\tanh u = \sec h^2 u \frac{du}{dx}$$

The student should be familiar with the employment of chain rule in the derivations. These derivatives can be easily achieved by normal differentiating after expressing the functions in their exponential form. For example,

$$\frac{d}{dx}\sinh x = \frac{1}{2}\frac{d}{dx}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} + e^{-x})$$
$$\frac{d}{dx}\sinh x = \cosh x$$

We thus conveniently turn these around to get their integrals namely

$$\int \cosh x \, dx = \sinh x + c$$
$$\int \sinh x \, dx = \cosh x + c$$

And also

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{d(\cosh x)}{\cosh x} = \ln(\cosh x) + C$$

Now, let us look at the inverse of these hyperbolic functions. It so turns out that from these inverses we can get some useful integration results. From the graph of the hyperbolic sinh, we can assume that there exist an inverse,

$$y = \sinh^{-1} x$$

which is obtained by solving

$$x = \sinh y = \tfrac{1}{2}(e^{y} - e^{-y})$$

for y in terms of x. By some rearranging and multiplying throughout by e^{y}, we get

$$\begin{aligned} &2x = e^{y} - e^{-y} \\ &e^{2y} - 2xe^{y} - 1 = 0 \end{aligned}$$

From here we realize it is a quadratic equation in term so of e^{y}, and by solving it we have

$$\begin{aligned} e^{y} &= \frac{2x \pm \sqrt{4x^{2} + 4}}{2} \\ &= x + \sqrt{x^{2} + 1} \end{aligned}$$

where we discard the minus square root because there are no negative values for e^{y}. By solving for y, we now obtain

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^{2} + 1})$$

The derivative of *y* can be found by differentiating this equation. However, I would suggest implicit differentiation as it simplifies the algebra. From,

$$\sinh y = x$$

We differentiate w.r.t *x* to get,

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

On using the identity $\cosh^2 y - \sinh^2 y = 1$. Here is where we get a new and useful integral result. Rearranging the above we have

$$\frac{dx}{\sqrt{1+x^2}} = dy$$

Now integrating both sides,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int dy = y + c$$

and then substituting back our original equation of y we yield

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + c$$

This result is highly useful in solving integrals of that form. In addition, we will use this result to solve the classical problem of finding the exact equation of a catenary: the shape of a flexible chain of uniform density which is suspended between two points and hangs under its own weight.