Calculus

Kepler's First Law

We resume by deriving Kepler's First law which says that a planet revolves in an elliptical orbit with the sun at one of its focus. This simple looking law actually turns out to be the hardest one to derive.

Again I advise careful reading of the mathematics. So get your differentiation techniques under your fingertips and let's see how it is done. (Scroll down for a diagram to assist in the explanations.)

We start by first examining Newton's inverse square law of universal gravitation. We know that $\vec{\mathbf{F}}$ is a central attractive force between *m* and *M* and is given by

$$F_r = -G\frac{Mm}{r^2}$$

Knowing that *G*, the gravitational constant, and *M*, the mass of the Sun, are constant, we simplify the algebra by writing

$$F_r = -\frac{km}{r^2}$$

where $k = GM$, and our second equation of motion becomes,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}$$

So how do we proceed with this complicated differential equation? Here's the plan. Now remember that we seek to find an equation of the orbit, possibly one written in polar form. Here are a few goals to look at:

1. We wish to somehow get $r = f(\theta)$, the equation of the orbit.
2. We need to eliminate t .
3. Most probably r will be a dependent variable and θ an independent one.

Now let's get started. Remember the small equation we had from the previous section, namely

$$r^2 \frac{d\theta}{dt} = h$$

We will substitute this into our differential equation to get

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}$$

The presence of r to a negative power suggest that it might be temporarily convenient to introduce a new dependent variable like $z = 1/r$. Now let's see. To eliminate *t,* it makes sense to express d^2r/dt^2 in terms of $d^2z/d\theta^2$ through some hard work of differentiation.

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{d\theta}\frac{d\theta}{dt}$$
$$= -\frac{1}{z^2}\frac{dz}{d\theta}\frac{h}{r^2} = -h\frac{dz}{d\theta}$$

Using our small equation again and differentiating a second time,

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{dz}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{dz}{d\theta}\right)\frac{d\theta}{dt}$$
$$= -h\frac{d^2z}{d\theta^2}\frac{h}{r^2} = -h^2z^2\frac{d^2z}{d\theta^2}$$

We make this substitution into our original differential equation to yield

$$-h^2z^2\frac{d^2z}{d\theta^2} - h^2z^3 = -kz^2$$

and after simplifying

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2}$$

With *t* gone, the equation looks more manageable. We're moving progress indeed. We further notice that, except the constant term on the right, this is a differential equation of simple harmonic motion where the acceleration is proportional to the displacement in the opposite direction. We simply put

$$w = z - \frac{k}{h^2}$$

With $d^2w/d\theta^2 = d^2z/d\theta^2$ and so

$$\frac{d^2w}{d\theta^2} + w = 0$$

giving us the general solution of this equation as

$$w = A\sin\theta + B\sin\theta$$

and so

$$z = A\sin\theta + B\sin\theta + \frac{k}{h^2}$$

We find a particular solution by using the following reasoning. With reference to the diagram at the bottom, we shift the direction of the polar axis in such a way that *r* is minimal implying *m* is closest to the origin. This occurs at $\theta = 0$. So by our equation $z = 1/r$, this means *z* to be a maximum in this direction, so

$$\frac{dz}{d\theta} = 0 \text{ and } \frac{d^2z}{d\theta^2} < 0$$

when $\theta = 0$. Differentiating the equation in *z* once and twice through and letting $\theta = 0$, we equate the coefficients *A* and *B* to get

$$A = 0 \text{ and } B > 0$$

Replacing *z* and *r,* we finally get the equation we intended.

$$r = \frac{1}{k/h^2 + B\cos\theta} = \frac{k/h^2}{1 + \left(Bk/h^2\right)\cos\theta}$$

All left to be done is to recognize that this is an equation of an ellipse. We put $e = {Bh^2}/{k}$ giving us the equation of the orbit as

$$r = \frac{h^2/k}{1 + e\cos\theta}$$

where *e* is a positive constant.

What do we know about conic sections? The above represents the polar equation of a conic section with focus at the origin and that this conic section is an ellipse, a parabola, or a hyperbola when $e < 1$, $e = 1$, or $e > 1$. By logical reasoning, since the planets remain in the solar system and do not move infinitely far away from the sun, the ellipse is the only possibility.

While the actual calculation *e* still needs to be done by taking an example of the earth's orbit, our deduction is sufficient to prove Kepler's first law, namely the orbit of each planet is an ellipse with the sun at one focus.

ERRATA: In the video, I made the wrong substitution of $e = {h^2}/{k}$ forgetting about the B term. Please accept my apologies. The equation here is the correct one. Nevertheless, the equations of conic sections still apply for both.

The same diagram has been place here for easy reference.