Calculus

Kepler's Third Law

We wrap up our section of Kepler's Laws by proving his final law, which states that the *square* of the period of revolution of a planet is proportional to the *cube* of the semimajor axis of the planet's elliptical orbit.

While this section isn't as algebra heavy as the previous one, a sound knowledge of elliptic curves would be useful.

Now that we have proven the case that *m* has an elliptic orbit, whose polar equation is

$$r = \frac{h^2/k}{1 + e\cos\theta}$$

we can immediately write the rectangular equations as

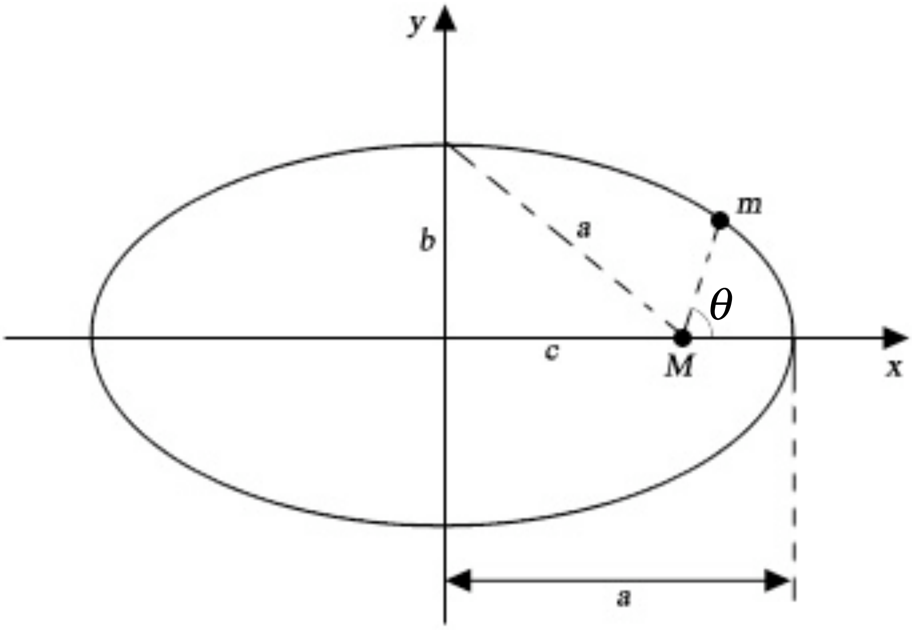
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here is where some work elliptic curve is needed. For an elliptic curve which takes the form of the give polar equation, the value of e, termed *eccentricity*, describes the shape of the ellipse.

Incidentally, this value can also be written as

$$e = \frac{c}{a}$$

where *c* and *a* corresponding the those constants in the rectangular equation above. Graphically, the curve is shown as



Now, some geometric observation gives us

$$c^2 = a^2 - b^2$$

and so

$$e^2 = (a^2 - b^2)/a^2$$
$$b^2 = a^2(1 - e^2)$$

Time for some astronomy knowledge. The semimajor axis *a* of an elliptical orbit is also called the *mean distance* because it is one-half the sum of the least and greatest values of *r*. These values of *r* correspond to $\theta = 0$ and $\theta = \pi$. By this definition, we substitute these values of θ into the polar equation to get the corresponding values of *r* to calculate *a.*

$$a = \frac{1}{2}\left(r\big|_{\theta=0} + r\big|_{\theta=\pi}\right) = \frac{1}{2}\left(\frac{h^2/k}{1+e} + \frac{h^2/k}{1-e}\right)$$
$$= \frac{h^2}{k(1-e^2)} = \frac{h^2a^2}{kb^2}$$

On rearranging, we get

$$b^2 = \frac{h^2 a}{k}$$

We now define *T* as the period of *m*, that is the time require for one complete revolution in its orbit. We also take the area of the ellipse to be πab and recalling Kepler's second law as $A(t_2) - A(t_1) = \frac{1}{2}h(t_2 - t_1)$, we have

$$T = \frac{2\pi ab}{h}$$

And so

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \left(\frac{4\pi^2}{k}\right)a^3$$

on eliminating *b* and *h*. We previously defined $k = GM$, which depends on the gravitational constant and mass of the Sun, which are both constants. Hence for any planet rotation around our solar system they obey Kepler's third law: The squares of the period of revolution of the planet is proportional to the cubes of their mean distances.

I hope you have enjoyed these 5 lessons on deriving Kepler's three laws. As a conclusion I would like to say my opinion on the value of mathematics. What I have done is perhaps one of the more mathematical rigorous approaches in deriving the laws. Whether this proof surpasses that of using principles in physics, it is left for the student to decide. However, what I can safely say is that fully integrating math in the proof does somewhat imply a greater deal of precision. Under whatever given assumptions, the equations don't lie in math and once we can establish a law in terms of variables and numbers, the law will hold forever.