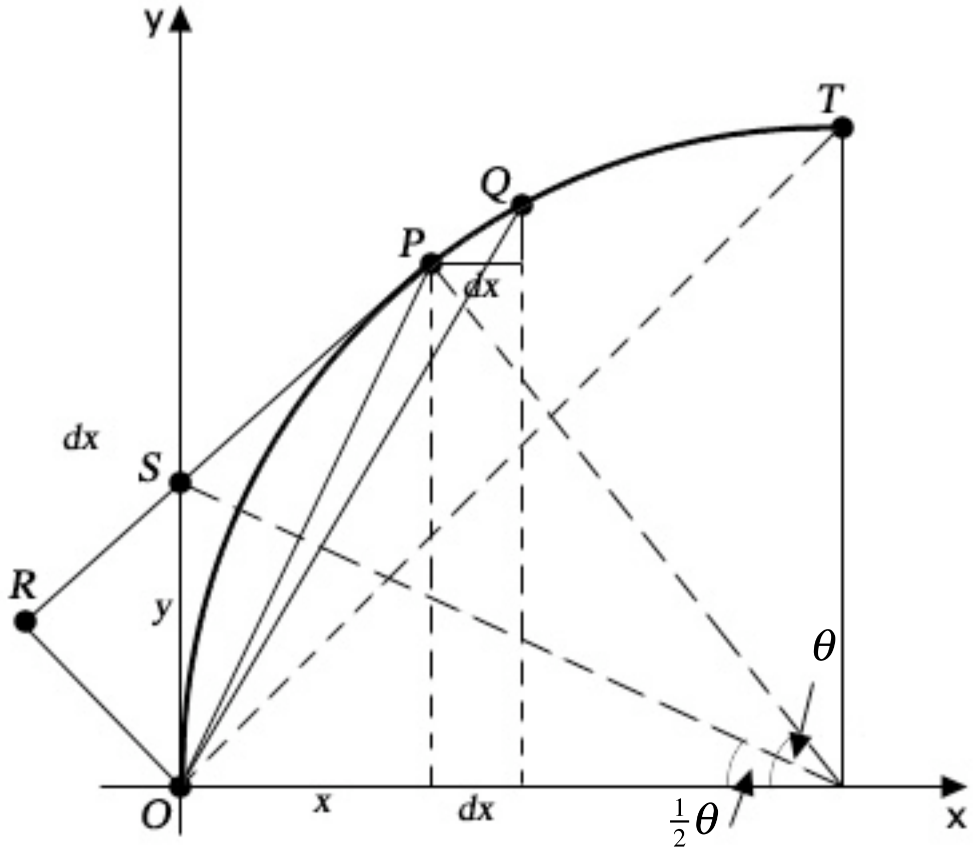
Calculus

## Leibniz's Quest for Pi

We all know that the area of a quarter of a circle of radius 1 is $\pi/4$. However, I bet you didn't know that apart from using geometric tools, there is another more ingenious and astounding way of calculating $\pi/4$. A method so great that it combines integration, trigonometry and series expansion and the man responsible for it, Newton's arch rival Leibniz. Let's see how he did it.

The area we are concern with is area *A* of the circular segment cut off by the chord *OT* as illustrated below. By finding this area, we simply add the area of the triangle which is $1/2$ to get the area of a quarter of a radius 1 circle.

While you may be busy trying to find how we will apply the Riemann sums method in finding the area, here is what Leibniz did. He considered the area of *OPQ*, the rather sliverlike elements whose base is the segment *PQ* of length *ds*. To get the area of a single element, we extend *PQ* to the area around *O* and then drop a perpendicular from *PQ* at *R* to meet *O*, thus *OR* represents the height of triangle *OPQ*. A graceful play of geometry indeed.



From the diagram, we can see that the two similar triangles gives us

$$\frac{ds}{dx} = \frac{OS}{OR} \text{ and thus } OS\,ds = OS\,dx$$

(For a quick explanation, just shift the angle at *P* to the y-axis and you'll see that this is the same angle at *O* of triangle *ORS*, bearing in mind that *R* is a right angle.)

We now represent a small area of sliver element *OPQ* as *dA* giving us

$$dA = \tfrac{1}{2} OR\,ds = \tfrac{1}{2} OS\,dx = \tfrac{1}{2}\, y\,dx$$

Here, *y* denotes the length of the segment *OS*. The element of area *dA* sweeps across the circular segment as *x* increase from 0 to 1, what is initially wanted to find. So,

$$A = \int dA = \frac{1}{2}\int_0^1 y\,dx$$

Anticipating that it will be easier to integrate a function of x w.r.t y, we use integration by parts to switch the variables.

$$A = \int_0^1 y \frac{d}{dx} x\,dx$$

$$= \frac{1}{2} xy\Big|_0^1 - \frac{1}{2}\int_0^1 x\,dy = \frac{1}{2} - \frac{1}{2}\int_0^1 x\,dy$$

where the limits of integrals are $y = 0$ and $y = 1$.

Now it is here where we seem to be stuck. The idea is that we must find an equation of *x* in terms of *y* so that we can do the necessary integration. This poses a difficult problem as *y* is hard to define algebraically. Here comes Leibniz with his idea of using trigonometry. From the diagram, notice that

$$y = \tan\theta \text{ and } x = 1 - \cos\theta = 2\sin^2 \tfrac{1}{2}\theta$$

(If you need convincing of the angles $\theta$ and $\frac{1}{2}\theta$, just observe that the angles at *P* and *O* are both right angles and so a straight line bisects that ‘kite’ shape.)

We now employ some trigonometry identities and write,

$$\tan^2 \tfrac{1}{2}\theta = \frac{\sin^2 \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta} = \sin^2 \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta$$
$$= \sin^2 \tfrac{1}{2}\theta(1 + \tan^2 \tfrac{1}{2}\theta)$$

which nicely gives us

$$\frac{x}{2} = \frac{y^2}{1+y^2}$$

our key in doing the integration.

But wait, there’s still a little more work to be done. Recall a little formula that enables us to expand as an infinite geometric series.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

which allows us to write

$$\frac{x}{2} = y^2(1 - y^2 + y^4 - y^6 + \ldots) = y^2 - y^4 + y^6 - y^8 + \ldots$$

and substituting this into our calculation of the area *A*,

$$\begin{aligned} A &= \frac{1}{2} - \int_0^1 y^2 - y^4 + y^6 - y^8 + \dots dy \\ &= \frac{1}{2} - \left[ \frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{7}y^7 - \frac{1}{9}y^9 + \dots \right]_0^1 \\ &= \frac{1}{2} - \left( \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots \right) \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \end{aligned}$$

We add $\frac{1}{2}$ to this area to account for the quarter triangle of the circle. The result is equated to the known area of $\pi/4$ and so we have Leibniz's formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

which is a fine testament of a great mathematician linking different branches of mathematics together to find an irrational number. With this discovery, Leibniz undoubtedly won the battle over Newton in this area.