

Calculus

Wallis's Product

In this lesson, we will spend some time looking at an ingenious way of expressing $\frac{\pi}{2}$ as an infinite product. English mathematician John Wallis discovered this remarkable expression in 1656 and it's called Wallis's product.

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \frac{2n}{2n+1} \cdots$$

As interesting as this sequence stands on itself, it has a place in important developments in both pure and applied mathematics. We shall prove the product here.

Proving Wallis's product involves 3 steps, and it uses integral calculus and limits in doing so. The steps are – establishing a reduction formula, using the reduction formula, and taking the limit.

Step 1 – Establishing the Reduction Formula

The reduction formula we will be using commonly appears in a typical calculus course. It is

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

While I do not intend to talk more about the reduction formula, we shall quickly show the above. Think of it as a recap for calculus students.

$$\text{Let } I_n = \int \sin^n x dx.$$

We then express the integral as

$$I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx$$

In doing so, we can express the integral as a product and then integrate by parts.

$$\begin{aligned}
 \int \sin^{n-1} x \sin x dx &= \int \sin^{n-1} x \frac{d}{dx}(-\cos x) dx \\
 &= -\sin^{n-1} x \cos x - \int -\cos x (n-1) \sin^{n-2} x \cos x dx \\
 &= -\sin^{n-1} x \cos x + \int \cos^2 x (n-1) \sin^{n-2} x dx \\
 &= -\sin^{n-1} x \cos x + \int (1 - \sin^2 x)(n-1) \sin^{n-2} x dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx
 \end{aligned}$$

We recognize here that the term $(n-1) \int \sin^n x dx$ is actually $(n-1)I_n$ and so can be brought over to the left hand side.

$$\begin{aligned}
 I_n + (n-1)I_n &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx \\
 nI_n &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx \\
 I_n &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x dx
 \end{aligned}$$

Thus establishing the reduction formula we needed.

Step 2 - Using the Reduction Formula

First, limits need to be placed in the integral defined by I to eliminate the evaluated part in the integration and to bring in the number π into the equation. And so we redefine

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$$

I suspect 0 and $\frac{\pi}{2}$ was chosen to eliminate $-\frac{1}{n} \sin^{n-1} x \cos x$, as when evaluated will yield 0 for one of the trigonometry functions. It's good to know such small details. We can express the previous result as

$$I_n = \frac{n-1}{n} I_{n-2}$$

for easier manipulation. We now find I_0 and I_1 which is clearly

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \text{ and } I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$$

and the $\frac{\pi}{2}$ has entered the picture. Since the n in the I_n can take integer values, we want to distinguish the cases of even and odd subscripts anticipating that different sequences can be formed for each case.

If the subscript is odd, we have

$$\begin{aligned} I_{2n} &= \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} I_{2n-4} \\ &= \dots = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \dots \frac{1}{2} I_0 \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2n-1}{2n} \cdot \frac{\pi}{2} \end{aligned}$$

What we did was to keep subtracting 2 from the subscript to reduce it to 0. Of course, n can take any integer value but we'll write it out as a sequence for whatever n . The last step involves rearranging the order but leaving the pi term at the back.

If the subscript is odd, we have

$$\begin{aligned} I_{2n+1} &= \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3} \\ &= \dots = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \dots \frac{2}{3} I_1 \\ &= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \dots \frac{2n}{2n+1} \end{aligned}$$

working along the lines of the previous question. With these two sequences, we can now move to the last step.

Step 3 - Taking the Limit

In this chain of reason, we need the fact that the ratio of these two quantities approaches 1 as n tends towards ∞ equivalent of saying,

$$\frac{I_{2n}}{I_{2n+1}} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

First notice the boundaries of the values set for the definite integral and the corresponding range of the $\sin x$ function.

$$0 \leq x \leq \frac{\pi}{2}$$
$$0 \leq \sin x \leq 1$$

Knowing that by raising the power of a base which is less than 1, we get a smaller number, we can write the following inequality.

$$0 \leq \sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x$$

Which in turn implies

$$0 < \int_0^{\frac{\pi}{2}} \sin^{2n+2} x dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n} x dx$$

or equivalently,

$$0 < I_{2n+2} \leq I_{2n+1} \leq I_{2n}$$

We now divide through by I_{2n} to get

$$\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

But from our reduction formula, we have the result,

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

which yields,

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

Now, as $n \rightarrow \infty$, the middle term gets smacked between both '1's to the left and the right and so we can say that

$$\frac{I_{2n+1}}{I_{2n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

which is the same as our initial limit.

We now start forming infinite product and with the limit, get the intended result. From our previous sequences of even and odd subscripts, dividing I_{2n+1} by I_{2n} gives us

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi}$$

on rearranging

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{I_{2n}}{I_{2n+1}}$$

On taking the limit as $n \rightarrow \infty$, we have

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

which is the beautiful formula known as Wallis product. I really marvel at the ingenious thinking that goes behind forming this expression. Who would have know that π , far behind any geometry use, can be expressed as an infinite product.