Complex Numbers 1977 AHSME #16

What is the value of $\sum_{n=0}^{40} i^n \cos(45+90n)^\circ$?

This is one of the more difficult question found in the AMC. It borrows concepts from Complex numbers and summation to solve it. As always, look for a repeating pattern from the first few terms and see whether they have a common sum, in this case, the first 4 terms.

By evaluating the first four terms n=0 to n=3, we have:

$$\text{when } n=0, i^0=1, \cos(45)=\frac{1}{\sqrt{2}}$$

$$\text{when } n=1, i^1=i, \cos(135)=-\frac{1}{\sqrt{2}}$$

$$\text{when } n=2, i^2=-1, \cos(225)=-\frac{1}{\sqrt{2}}$$

$$\text{when } n=3, i^3=-i, \cos(315)=\frac{1}{\sqrt{2}}$$

On first sight, we immediately see a pattern. Due to a nature of the imaginary number i and the cosine function, there a high chance that terms will repeat themselves and thus they do.

$$\text{when } n=0, a_0=i^0 \text{x} \cos(45)=\frac{1}{\sqrt{2}}$$

$$\text{when } n=1, a_1=i^1 \text{x} \cos(135)=-\frac{i}{\sqrt{2}}$$

$$\text{when } n=2, a_2=i^2 \text{x} \cos(225)=\frac{1}{\sqrt{2}}$$

$$\text{when } n=3, a_3=i^3 \text{x} \cos(315)=-\frac{i}{\sqrt{2}}$$

At this point, I am tempted to suspect that the terms cancel each other out because for each i and cosine term, they positive and negative repeating pattern suggest likewise. However, beware that the question is asking for the multiplication of the two, as such.

And so we sum from n=0 to n=3 to see what we have.

$$\sum_{n=0}^{n=3} i^n \cos(45+90n)^\circ = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} - \frac{2i}{\sqrt{2}}$$

Knowing the sum from n=0 to n=3, recognize that the sum of terms n=4 to n=7 is the same that of n=0 to n=3. And the amount of these 'group of four' terms occurs 10 times up to the term n=39, be careful of the counting.

$$\sum_{n=0}^{n=39} i^n \cos(45+90n)^\circ = 10\text{x}\left(\frac{2}{\sqrt{2}} - \frac{2i}{\sqrt{2}}\right)$$
$$= \frac{20}{\sqrt{2}} - \frac{20i}{\sqrt{2}}$$

Don't forget the 40th term. Remember, n=0 to n=40 has 41 terms and so when we sum the 'group of four' terms 10 times, it brings up to the term 39th. We need to add the 40th term which is the same as the 0th term as shown previously.

$$\sum_{n=0}^{n=40} i^n \cos(45+90n)^\circ = \frac{20}{\sqrt{2}} - \frac{20i}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{21}{\sqrt{2}} - \frac{20i}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(21-20i)$$

Now you know how to solve such a problem!