Complex Numbers 1990 AHSME #22

The six solutions of $z^6 = -64$ are written in the form $a+bi$, where a and b are real numbers. What is the product of those solutions with a > 0?

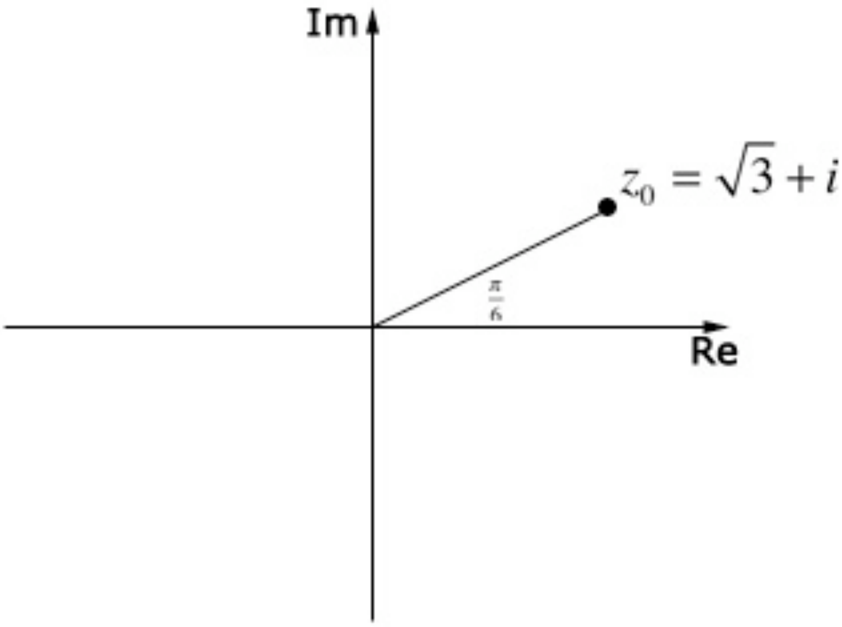
The main idea here is the application of De Moivre's formula. However, the small twist in this question is to recognize what is the given complex number. Comparing with the formula I previously presented which is,

when $w^n = z$, where $z$ is a given complex number, then

$$w_i = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

for each $k = 0,1,2,\ldots,n-1$.

We are very quick to think that the given complex number in this case is $z^6$. But this is certainly NOT the case. Look at the formula carefully, the z in the question refers to w and the given complex number is -64. The mistake students often make is in thinking that the complex must have an imaginary part. No, instead we need to express the number -64 in polar form so that De Moivre's theorem can be used. Drawing -64 on the complex plane,




the number lies on the real axis and so, by way of the magnitude and argument, we can write, $-64 = 64(\cos\pi + i\sin\pi)$

And so now we then apply De Moivre's theorem giving us,

$$w_i = 64^{\frac{1}{6}}\left(\cos\left(\frac{\pi+2k\pi}{n}\right)+i\sin\left(\frac{\pi+2k\pi}{n}\right)\right)$$

for $k = 0,1,2,3,4,$ and 5.

Since, we are looking for solution for which the real part, a, is more than 0, we simply substitute the values of k inside and see which ones are a > 0 for this solution. In doing so, this gives us,

$$z_0 = \sqrt{3}+i$$
$$z_1 = 2i$$
$$z_2 = -\sqrt{3}+i$$
$$z_3 = \sqrt{3}-i$$
$$z_4 = -2i$$
$$z_5 = \sqrt{3}-i$$

And finally, the product of the two complex numbers with a > 0 is,

$$z_0 \cdot z_5 = (\sqrt{3}+i)\cdot(\sqrt{3}-i) = 4$$