

## Complex Numbers

### **Addition and Multiplication**

Complex numbers questions will only show up in the AMC 12, so you need not worry if you are taking AMC 10. The reason for this is that high school students would have yet to learn complex numbers in their syllabus around that time. But that shouldn't prevent you from learning if you see it as interesting.

What I can say about complex numbers is that it is one of those subjects where a few definitions and principles span a whole range problems which will become solvable should you have the concepts in place. Just like how complex numbers are still numbers, I like to think of them as an extension to the real plane.

We shall now formally define complex numbers. The set of complex numbers, usually denoted by  $\mathbb{C}$ , consist of all expressions in the form

$$z = a + bi$$

where  $a$  and  $b$  are real and  $i = \sqrt{-1}$ . Written in formal notation, it would be

$$\mathbb{C} = \{z = a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$$

The number  $a$  is called the real part of  $z$  and  $b$  called the imaginary part of  $z$ . It follows that a complex number with no imaginary part, or  $b = 0$ , is called a real number.

Alternative ways of expressing the real and imaginary parts are

$$\operatorname{Re}(z) = a$$

$$\operatorname{Im}(z) = b$$

Later we shall see that there is what we call a **polar form** of a complex number. For now, a complex number in standard form is written as

$$(\text{real part}) + (\text{imaginary part})i$$

We shall now go into some definitions.

**Two complex numbers are equal if their respective real and imaginary parts are equal.**

If  $z_1 = a_1 + b_1i$  and  $z_2 = a_2 + b_2i$   
then  $z_1 = z_2$  if and only if  
 $a_1 = a_2$  and  $b_1 = b_2$

**The sum of two complex numbers is simply the sum of its respective real and imaginary parts.**

If  $z_1 = a_1 + b_1i$  and  $z_2 = a_2 + b_2i$   
then  $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$

This result is pretty self-explanatory when thinking along the lines of treating the real part and imaginary part separately when doing any form of algebra. The multiplication of 2 complex numbers is a little more confusing.

**The multiplication of two complex numbers is as follows.**

If  $z_1 = a_1 + b_1i$  and  $z_2 = a_2 + b_2i$   
then  $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i$

We get the above result by multiplying the individual real and imaginary parts of each number with the other one and then treating them separately, similar to how we multiply a linear polynomial.

If  $z_1 = a_1 + b_1i$  and  $z_2 = a_2 + b_2i$   
then  $z_1 \cdot z_2 = (a_1 + b_1i) \cdot (a_2 + b_2i)$   
 $= a_1a_2 + a_1b_2i + b_1ia_2 + b_1ib_2i$   
 $= (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i$

bearing in mind that  $i^2 = -1$ .

There you have it, the basic definitions of complex numbers. Next up, the **conjugate** of a complex number.