Complex Numbers

The Complex Plane – Addition and Multiplication of Complex Numbers

In the previous section, we ended up with Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$ which we describe as one of the more important formula, well if any formula can be said as unimportant, in mathematics. From Euler's Formula, it allows us to re-represent, or write the complex number in another form.

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

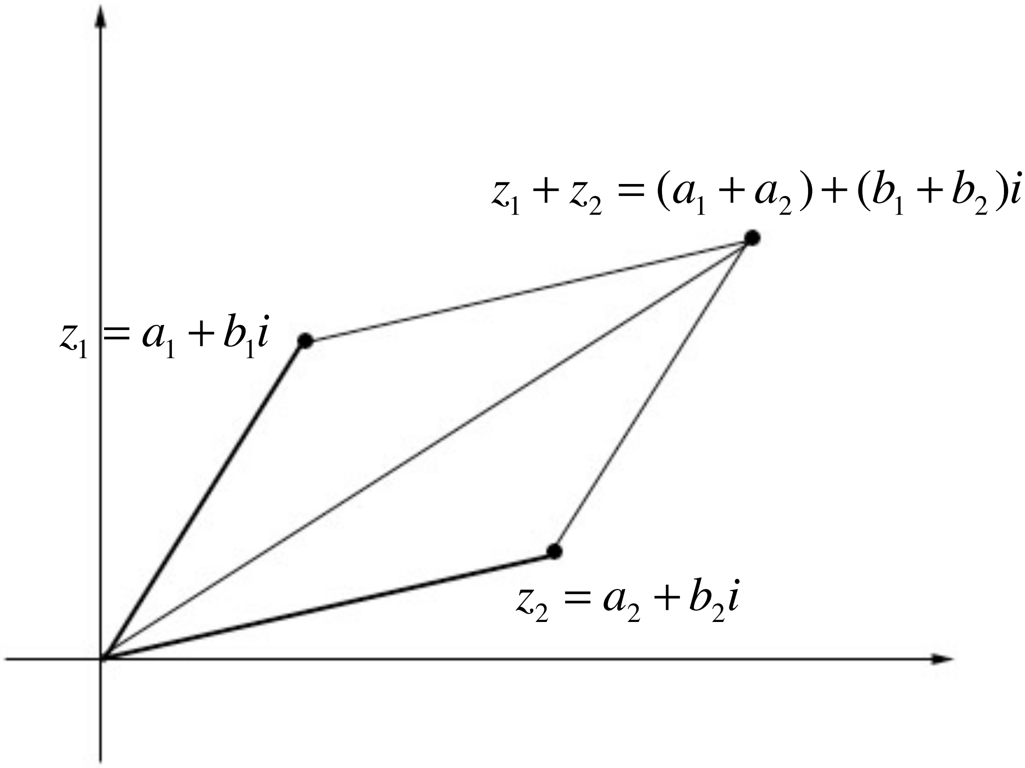
to which we enforce again that $r = |z|$, the magnitude of z and the argument is θ.

Drawing from Euler's Formula and the laws of indices we can find another way to rewrite the complex number e^{x+iy}, which is

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

This becomes interesting because notice that the magnitude of the complex number e^{x+iy} is only dependent on x as $r = e^x$.

Since we have developed ideas of the complex plane, we can now geometrically represent the addition of two complex numbers. Suppose we have the complex numbers $z_1 = a_1 + b_1 i \text{ and } z_2 = a_2 + b_2 i$. The addition of the two complex numbers can be thought of as vector addition. The values of a are along the real axis and the values of b are along the imaginary axis.



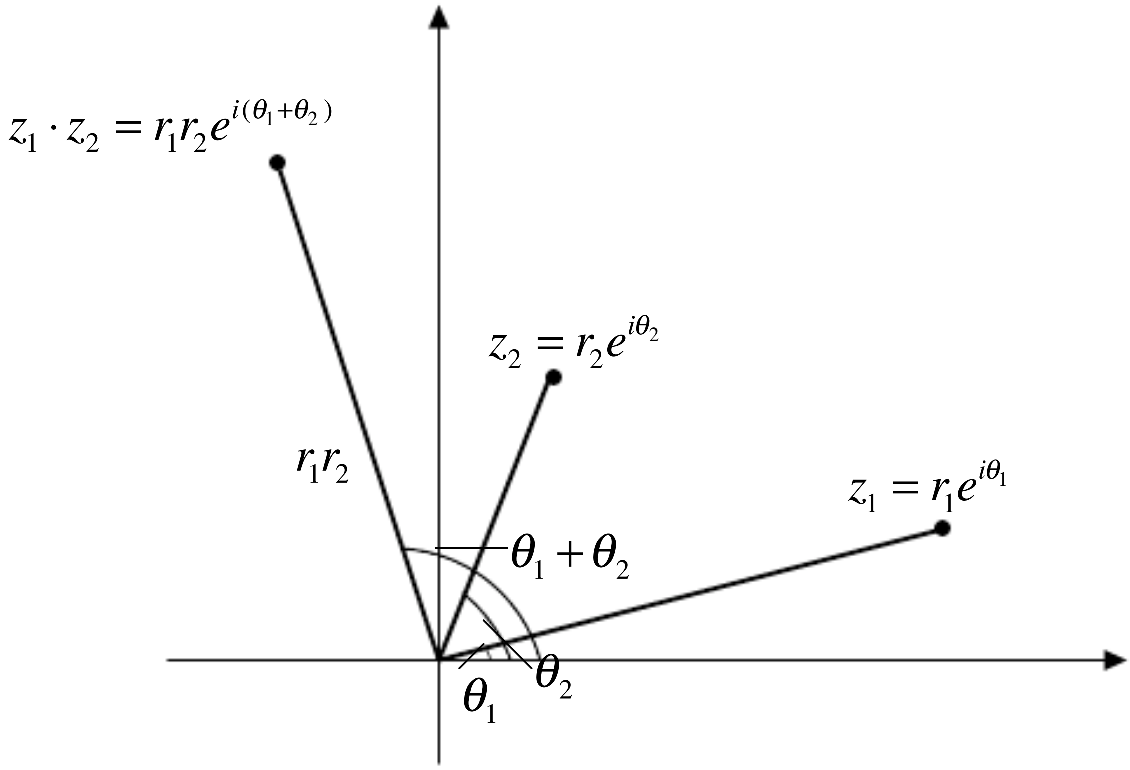
While geometrical representation of addition is fairly obvious, the process of multiplying two complex numbers, at least in standard form, does not easily give us a geometrical representation. In order find the meaning of multiplying two complex numbers in the complex plane, we first convert them using Euler's formula as previously shown.

$$z_1 = r_1 e^{i\theta} \text{ and } z_2 = r_2 e^{i\theta}$$

where $r_1, \theta_1 \text{ and } r_2, \theta_2$ are the magnitudes and argument respectively. Then by multiplying both together applying the laws of indices, we have

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right) \end{aligned}$$

This shows that the product of $z_1 \text{ and } z_2$ is found by multiplying the magnitudes and adding the arguments, as illustrated below.



We conclude by using Euler's Formula to rewrite the reciprocal of a complex number.

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

which should be fairly straight forward.

In this chapter, we have learnt three different ways to represent a complex number them being the standard form, polar form and Euler's form (though I don't think that is what it is called)

We have also found geometrical meanings of the addition and multiplication of complex numbers in the complex plane.