Complex Numbers
de Moivre's theorem

We can now prove de Moivre's theorem, a theorem needed when finding the nth roots of a complex number as well as being an indispensable tool when dealing with complex numbers.

de Moivre's theorem states that for any integer n and any real number θ,

$$\left[\cos(\theta)+i\sin(\theta)\right]^n = \cos(n\theta)+i\sin(n\theta)$$

de Moivre's theorem can be prove using Euler's formula, which is a easier way or by using trigonometry function and the results we showed in the previous lessons. I shall show you both ways.

This result follows naturally from Euler's Formula.

$$\left[\cos(\theta_1)+i\sin(\theta_1)\right]^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta)+i\sin(n\theta)$$

While this is all good, I'm sure many of you mathematicians out there, well at least for me, would like a challenge of proving it without Euler's Formula. Read on.

Formally stated, we are to prove

$$\left[\cos(\theta)+i\sin(\theta)\right]^n = \cos(n\theta)+i\sin(n\theta) \quad \forall n \subset \mathbb{Z}$$

Clearly the proposed equation is true if n=0. Now let n be any positive integer, i.e. $n \subset \mathbb{Z}^+$. Let $z = \cos(\theta)+i\sin(\theta)$. Then

$$\left|z^n\right| = \left|z\right|^n = 1^n = 1$$

and

$$\arg(z^n) = n\arg(z) = n\theta$$

And so the polar form of z^n is $\cos(n\theta)+i\sin(n\theta)$. Therefore,

$$z^n = \left[\cos(\theta) + i\sin(\theta)\right]^n = \cos(n\theta) + i\sin(n\theta) \quad \forall n \subset \mathbb{Z}^+$$

This completes the proof if n is a nonnegative integer.

Now suppose that n is a negative integer. Then, by introducing a new term, we can say $m = -n$ is a positive integer.

$$\begin{aligned}
\left[\cos(\theta) + i\sin(\theta)\right]^n &= \frac{1}{\left[\cos(\theta) + i\sin(\theta)\right]^m} \\
&= \frac{1}{\cos(m\theta) + i\sin(m\theta)} \\
&= \frac{1}{\cos(m\theta) + i\sin(m\theta)} \cdot \frac{\cos(m\theta) - i\sin(m\theta)}{\cos(m\theta) - i\sin(m\theta)} \\
&= \cos(m\theta) - i\sin(m\theta) \\
&= \cos(-n\theta) - i\sin(-n\theta) \\
&= \cos(n\theta) + i\sin(n\theta)
\end{aligned}$$

In the second line, we have used the first half of the proof since we know that m is nonnegative.

It is good to note that we have used two techniques here. One, we *rationalized the denominator* multiplying through with a term that removes the imaginary part. Two, we used a mixture of trigonometry identities such as $\cos^2\theta + \sin^2\theta = 1$ and $\cos(-\theta) = \cos(\theta), \sin(-\theta) = -\sin(\theta)$ to get the intended result.

It pays to notice how other branches of mathematics are used in this proof. We will now move to a direct application of de Moivre's theorem, find the nth roots of a complex number.