

Complex Numbers

**Nth Roots of a Complex Number**

With the help of de Moivre's theorem, we can find the nth roots of a complex number. Before even touching complex numbers, we should first establish what the nth root of a number means. The nth root of a number  $z$  is represented by

$$\sqrt[n]{z}$$

Perhaps we are all familiar with the square root  $\sqrt{z}$ , which in this term would be the 2<sup>nd</sup> root, and that with the result we get, by squaring it will give us  $z$ . Taking the nth root of  $z$  is similar, but it gives us a result to which we need to increase it to the power of  $n$  to get back  $z$ . Thus,

$$\sqrt[n]{z} = b$$

$$z = b^n$$

which should be easier for the reader to understand. Now relating it to complex numbers, when we are finding the nth root of complex number  $z$ , we are essentially finding a number such that by increasing it to the power of  $n$ , we get back  $z$ .

We shall now formally state the principle.

Suppose that  $n$  is a positive integer and  $z$  is a given complex number. There are  $n$  distinct  $n$ th roots of  $z$ , which are defined by

$$w_i = r^{\frac{1}{n}} \left( \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right)$$

for each  $k = 0, 1, \dots, n - 1$ .

Now by simply looking at this formula, we should inspect on some of its features. First, when we want to find the nth roots of  $z$ , we will get  $n$  roots, or simply put, when finding the 5<sup>th</sup> roots of  $z$ , we will get five different numbers that when each are individually taken to the power of five, we get back  $z$ . Quite interesting I must add because it seems that there are five numbers we can use to get 'back' to  $z$  by multiplying with itself. Second, the nth roots are distinct so no two are the same. Third and quite

importantly, the roots can be real or imaginary as a value of  $k$  might give  $\sin=0$ . I hope you see that.

To show this principle, we first write  $z$  and  $w$  in polar form, which is

$$z = r(\cos \theta + i \sin \theta) \text{ and } w = s(\cos \phi + i \sin \phi)$$

where  $r \geq 0$  and  $s \geq 0$ . By definition,

$$\sqrt[n]{z} = w$$

$$z = w^n$$

$$\begin{aligned} r(\cos \theta + i \sin \theta) &= s^n (\cos \phi + i \sin \phi)^n \\ &= s^n (\cos n\phi + i \sin n\phi) \end{aligned}$$

paying close attention to the application of de Moivre's theorem on the last step. By equating the magnitudes and argument, we have

$$s = r^{\frac{1}{n}}$$

and

$$\cos \theta = \cos n\phi$$

$$n\phi = \theta + 2k\pi, \text{ for some integer } k$$

Recognizing the periodicity of the cosine function. This gives  $n$  distinct  $n$ th roots of  $z$ .

Now, must of you might think why did we set the range of values that  $k$  can take to be from 0 to  $n-1$ . Well, to show that, we need to consider the geometric representation of these  $n$ th roots, which is the next lesson.