Complex Numbers

Proof of results using Trigonometric Identities

At this point, we have a good number of results, some of which was attained with the help of Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$. By using this formula, we can multiple complex numbers and in turn find out their magnitude and argument using laws of indices such as

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ &= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \end{aligned}$$

which says that the product of $z_1 \text{ and } z_2$ will give a resulting complex number which has a magnitude equal to the product of each number's magnitude and an argument which is the sum of each number's argument.

However, let's just say that for the moment, we need to prove the above results *without* Euler's Formula. Honestly, I do not know what result came first but having another way to show the product $z_1 z_2$ will undoubtedly be useful. And that is achieve using trigonometry identities.

We specify two complex numbers

$$z_1 = r_1 \left[\cos(\theta_1) + i\sin(\theta_1)\right]$$

and

$$z_2 = r_2 \left[\cos(\theta_2) + i\sin(\theta_2)\right]$$

then

$$\begin{aligned} z_1 z_2 &= r_1\left[\cos(\theta_1)+i\sin(\theta_1)\right] r_2\left[\cos(\theta_2)+i\sin(\theta_2)\right] \\ &= r_1 r_2\left[\cos(\theta_1)+i\sin(\theta_1)\right]\left[\cos(\theta_2)+i\sin(\theta_2)\right] \\ &= r_1 r_2 \left\{ \begin{array}{l} \left[\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\right] \\ +i\left[\sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)\right] \end{array} \right\} \end{aligned}$$

bearing in mind multiplying each term with the other two

$$= r_1 r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

using the trigonometry addition formula

This gives us the polar form of $z_1 z_2$ which we can immediately conclude that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

In addition, by looking at the angle inside the cosine and sine functions, we can see that, $\theta_1+\theta_2$, the sum of the argument of z_1 and the argument of z_2, is the argument of $z_1 z_2$.

A similar method can be used to calculate the magnitude and argument of the complex number $\frac{z_1}{z_2}$, which I will leave to the reader. You simply multiply the trigonometry identities as follows but use another addition formula to achieve the desired result.

I would like to conclude with proper notation to write the results.

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1 z_2) = \theta_1+\theta_2 = \arg(z_1)+\arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1-\theta_2 = \arg(z_1)-\arg(z_2)$$

It shouldn't take long for the reader to draw parallels with these results and the logarithm function.