Differential Vector Calculus
Curvature

Now that we have familiarize ourselves with the terms *velocity* and *acceleration* in 3 dimensional vectors, we will introduce a new term called *curvature*, and later another closely related one called *radius of curvature.*

We recap from the previous section that when

$$\vec{\mathbf{F}}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

we have

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{F}}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

I have previously shown that $\vec{\mathbf{F}}'(t)$ is in the direction of the tangent to the trajectory at the point $(x(t), y(t), z(t))$ defined by parameter *t*. I have then later shown that if we parameterize the curve in terms of its arc length *s* along the curve from some initial point, then $\vec{\mathbf{F}}'(s)$ has unit length and thus is the unit tangent vector. In cases where it is inconvenient to introduce the variable *s* as finding the equation $t = t(s)$ may prove difficult, we obtain the unit tangent vector by dividing $\vec{\mathbf{F}}'(t)$ by its length. And so we can then defined the *unit tangent vector* $\vec{\mathbf{T}}(t)$ as,

$$\vec{\mathbf{T}}(t) = \frac{1}{\left\|\vec{\mathbf{F}}'(t)\right\|}\vec{\mathbf{F}}'(t)$$

provided that $\vec{\mathbf{F}}'(t) \neq \mathbf{O}$. If $\vec{\mathbf{F}}'(t) = \mathbf{O}$, we do not define a tangent vector to the curve at $(x(t), y(t), z(t))$. If $t = s, \left\|\vec{\mathbf{F}}'(s)\right\| = 1$, and this equation gives us $\vec{\mathbf{T}}(s) = \vec{\mathbf{F}}'(s)$, which again is consistent with our previous definitions.

We will now define the new term. The *curvature* κ of a curve in three-space is the magnitude of the rate of change of the tangent vector with respect to *s,*

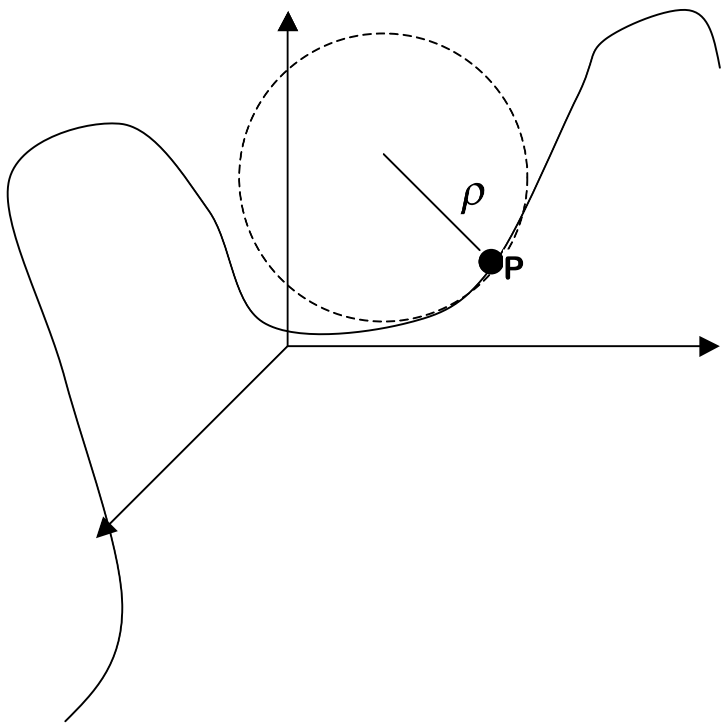
$$\kappa = \left\|\frac{d\vec{\mathbf{T}}}{ds}\right\|$$

The Greek symbol κ is pronounced as kappa. By looking at the equation, we suspect that the value of κ gives us the value of what we call the "amount of bending" or curvature. For example, the greater the magnitude of $\vec{\mathbf{T}}'(s)$, the more the curve bends for a given change in arc length. A straight line has a constant tangent vector and so it's curvature is 0. By using calculus, we can show that in terms of the position vector $\vec{\mathbf{F}}(t)$,

$$\kappa = \frac{\left\| \vec{\mathbf{F}}' \times \vec{\mathbf{F}}'' \right\|}{\left\| \vec{\mathbf{F}}' \right\|^3}$$

assuming that $\vec{\mathbf{F}}'(t) \neq \mathbf{O}$. We will rarely use the above equation.

Alongside the term curvature, we define another term. The quantity $\rho = \dfrac{1}{\kappa}$, the reciprocal of the curvature, is called the *radius of curvature* of a curve, provided that $\kappa \neq 0$. The Greek symbol ρ is called rho. It's value gives us the radius of a circle that best approximates the curve *C* at the point *P* on its concave side. Such a circle is called the *osculating circle* to the curve at *P*.



Next, we shall use the curvature and radius of curvature to define another two terms. The *unit normal vector* to the curve at the point *P* is given by

$$\vec{\mathbf{N}}(s) = \rho \frac{d\vec{\mathbf{T}}}{ds}$$

We can verify that this is a unit vector, by simply taking its magnitude like so

$$\left\|\vec{\mathbf{N}}\right\| = \rho\left\|\frac{d\vec{\mathbf{T}}}{ds}\right\| = \frac{1}{\kappa}\left\|\frac{d\vec{\mathbf{T}}}{ds}\right\| = 1$$

Morever, **N** is orthogonal to **T**. To prove this, we recall a basic property of the dot product which is the a vector dot by itself is equal to it's magnitude squared. Thinking along this lines and differentiating the equation, we get

$$\vec{\mathbf{T}}\cdot\vec{\mathbf{T}} = \left\|\vec{\mathbf{T}}\right\|^2 = 1$$

$$\frac{d}{ds}\left(\vec{\mathbf{T}}\cdot\vec{\mathbf{T}}\right) = \vec{\mathbf{T}}'(s)\cdot\vec{\mathbf{T}}(s) + \vec{\mathbf{T}}(s)\cdot\vec{\mathbf{T}}'(s) = \frac{d}{ds}(1) = 0$$

$$\vec{\mathbf{T}}'(s)\cdot\vec{\mathbf{T}}(s) = 0$$

But since $\vec{\mathbf{T}}(s)$ is orthogonal to $\vec{\mathbf{T}}'(s)$, and $\vec{\mathbf{N}}(s)$ is a positive scalar multiple of $\vec{\mathbf{T}}'(s)$ and so in the same direction as $\vec{\mathbf{T}}'(s)$, we conclude that $\vec{\mathbf{N}}(s)$ is orthogonal or perpendicular to $\vec{\mathbf{T}}'(s)$.

Next, we shall see how we can write the acceleration vector in terms of these two components, $\vec{\mathbf{N}}(s)$ and $\vec{\mathbf{T}}'(s)$.