Differential Vector Calculus

The Frenet-Serret formulas

Recall our previous results, that is

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau\vec{\mathbf{N}} \text{ and } \frac{d\vec{\mathbf{T}}}{ds} = \kappa\vec{\mathbf{N}}$$

It seems logical first that we find the first derivative of the unit normal $\vec{\mathbf{N}}$ with respect to arc length s. To get that, we simply represent $\vec{\mathbf{N}}$ in its cross product and differentiate accordingly, and employing substitution with the previous equations.

$$\begin{aligned}
\vec{\mathbf{N}} &= \vec{\mathbf{B}} \times \vec{\mathbf{T}} \\
\frac{d\vec{\mathbf{N}}}{ds} &= \frac{d}{ds}(\vec{\mathbf{B}} \times \vec{\mathbf{T}}) = \frac{d\vec{\mathbf{B}}}{ds} \times \vec{\mathbf{T}} + \vec{\mathbf{B}} \times \frac{d\vec{\mathbf{T}}}{ds} \\
&= -\tau\vec{\mathbf{N}} \times \vec{\mathbf{T}} + \kappa\vec{\mathbf{B}} \times \vec{\mathbf{N}} = -\kappa\vec{\mathbf{T}} + \tau\vec{\mathbf{B}}
\end{aligned}$$

Together the three formulas are

$$\frac{d\vec{\mathbf{T}}}{ds} = \kappa\vec{\mathbf{N}}$$

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau\vec{\mathbf{N}}$$

$$\frac{d\vec{\mathbf{N}}}{ds} = -\kappa\vec{\mathbf{T}} + \tau\vec{\mathbf{B}}$$

And so I present to you **The Frenet-Serret formulas** which are of fundamental importance in the theory of curve in 3-space. These formulas can also be represented in matrix form as:

$$\frac{d}{ds}\begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix}$$

We shall see very soon how a curve can be solely described by the functions of curvature $\kappa(s)$ and torsion $\tau(s)$ leading up to the fundamental theorem of space curves. This will get quite exciting!