

Differential Vector Calculus

The Fundamental Theorem of Space Curves

Let C_1 and C_2 be two non-vanishing curves, both of which has the same values of the curvature, $\kappa(s)$, and torsion, $\tau(s)$. The Fundamental theorem of space says that these two curves are congruent, meaning to say that one can be moved rigidly and then superimposed on the other so that they coincide exactly.

We shall prove this result.

At first we must set the condition $\kappa \neq 0$ as the vectors \vec{N} and \vec{B} are not defined when $\kappa = 0$. We then define the vectors $\vec{T}_1, \vec{N}_1, \vec{B}_1$ and $\vec{T}_2, \vec{N}_2, \vec{B}_2$ as the unit tangent, normal and binormal for the curves C_1 and C_2 respectively.

What we would do next is to define a scalar function $f(s)$ such that

$$f(s) = \vec{T}_1(s) \cdot \vec{T}_2(s) + \vec{N}_1(s) \cdot \vec{N}_2(s) + \vec{B}_1(s) \cdot \vec{B}_2(s)$$

What we anticipate is whether the above function would change as s varies. To find out, we differentiate $f(s)$ with respect to s using the Product Rule and the Frenet-Serret formulas. Long equation coming up. I have dropped the (s) to save space but vectors still mean the same.

$$\begin{aligned} f(s) &= \vec{T}'_1 \cdot \vec{T}_2 + \vec{T}_1 \cdot \vec{T}'_2 + \vec{N}'_1 \cdot \vec{N}_2 + \vec{N}_1 \cdot \vec{N}'_2 \\ &\quad + \vec{B}'_1 \cdot \vec{B}_2 + \vec{B}_1 \cdot \vec{B}'_2 \\ &= \kappa \vec{N}_1 \cdot \vec{T}_2 + \kappa \vec{T}_1 \cdot \vec{N}_2 \\ &\quad - \kappa \vec{T}_1 \cdot \vec{N}_2 + \tau \vec{B}_1 \cdot \vec{N}_2 - \kappa \vec{N}_1 \cdot \vec{T}_2 + \tau \vec{N}_1 \cdot \vec{B}_2 \\ &\quad - \tau \vec{N}_1 \cdot \vec{B}_2 - \tau \vec{B}_1 \cdot \vec{N}_2 \\ &= 0 \end{aligned}$$

Therefore, the function $f(s)$ is constant. We now move C_2 rigidly so that its initial point coincides with the initial point of C_1 and in a way 'twist' it so that the Frenet frames of both curves coincide at that point. Since the frames coincide at $s = 0$, the constant $f(s)$ must be 3 or

$$\vec{T}_1(s) \cdot \vec{T}_2(s) + \vec{N}_1(s) \cdot \vec{N}_2(s) + \vec{B}_1(s) \cdot \vec{B}_2(s) = 3$$

Bearing in mind that each vector is of unit length. However, we recall the definition of the dot product

$$\vec{T}_1 \cdot \vec{T}_2 = \|\vec{T}_1\| \|\vec{T}_2\| \cos \theta$$

which tells us that each dot product cannot exceed 1. Therefore, each dot product must equal to 1 as when one product is less than 1, the other products can't make the sum to 3.

In particular $\vec{T}_1(s) \cdot \vec{T}_2(s) = 1$ which implies that these vectors are equal for all s as this occurs when $\cos \theta = 1$ or $\theta = 0$ meaning both vectors the same direction and magnitude.

$$\frac{d\vec{R}_1}{ds} = \vec{T}_1(s) = \vec{T}_2(s) = \frac{d\vec{R}_2}{ds}$$

Integrating with respect to s and using the fact that both curves start from the same point when $s = 0$, we obtain

$$\vec{R}_1 = \vec{R}_2$$

for all s , concluding that C_1 and C_2 are congruent.

NOTE: In the video, I mentioned this to be a 'half proof' or that it is 'not a full proof'. Just to clarify, I never meant to discount or discredit the work of the mathematician involve in this. Instead, I was addressing the point of how the function $f(s)$ came about and should we know that, it would make lesson more complete. If anything, I am praising the great intuition in thinking of such a function.