Differential Vector Calculus

Multiple approaches to a problem

Using all that we know, we will now analyze a position vector of a curve and see the different approaches to get the vectors and quantities we want. Suppose the position vector of a particle at time $t > 0$ is

$$\vec{\mathbf{F}}(t) = [\cos(t) + t\sin(t)]\mathbf{i} + [\sin(t) - t\cos(t)]\mathbf{j} + t^2\mathbf{k}$$

Then

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{F}}'(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + 2t\mathbf{k}$$

$$\begin{aligned} v = \left\|\vec{\mathbf{v}}(t)\right\| &= \sqrt{t^2\cos^2(t) + t^2\sin^2(t) + 4t^2} \\ &= \sqrt{5t^2} = \sqrt{5}t \end{aligned}$$

We find the acceleration by

$$\vec{\mathbf{a}}(t) = [\cos(t) - t\sin(t)]\mathbf{i} + [\sin(t) + t\cos(t)]\mathbf{j} + 2\mathbf{k}$$

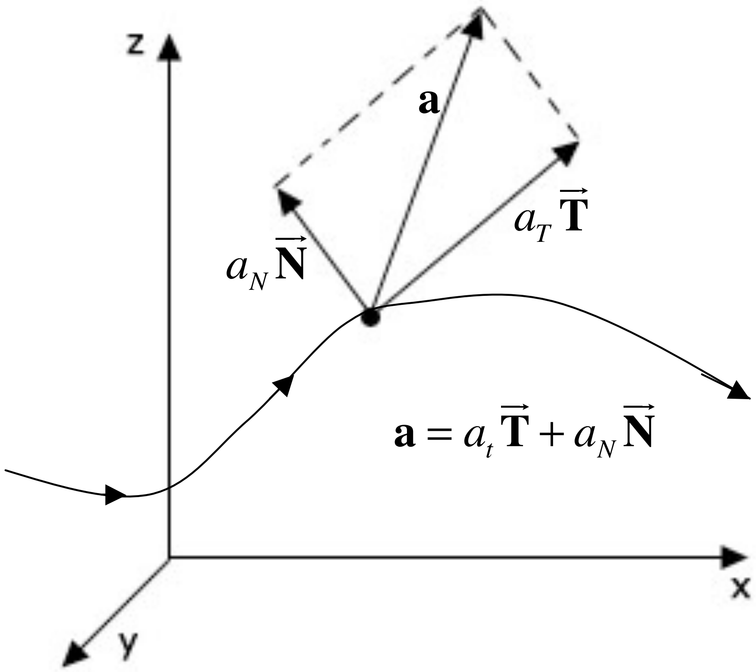
$$\begin{aligned} \left\|\vec{\mathbf{a}}(t)\right\| &= \sqrt{[\cos(t) - t\sin(t)]^2 + [\sin(t) + t\cos(t)]^2 + 4} \\ &= \sqrt{5 + t^2} \end{aligned}$$

Using our previous definitions, the tangential component of accleration is

$$a_T = \frac{dv}{dt} = \sqrt{5}$$

This is where we have multiple options to proceed with the problem. We are now concern with finding the normal component which is finding $\frac{1}{\rho}v^2$. This may prove tedious because we need the curvature which in turn means we need to the unit tangent vector. The problem comes by expressing the positive vector in terms of the arc length. Try to fit in $t = t(s)$ into the bunch of trigonometry functions and you'll get a headache. Nonetheless, not all is lost. We are fortunate to have another approach to the problem.

We first notice that the vectors **T** and **N** are perpendicular and so we apply the Pythagorean's theorem to the parallelogram sum $\vec{\mathbf{a}} = a_T \vec{\mathbf{T}} + a_N \vec{\mathbf{N}}$ as shown below.



Using the lengths a_N and a_T, we can find the magnitude, or the length of $\mathbf{a}$.

$$a^2{}_N = \|\mathbf{a}\|^2 - a^2{}_T = 5 + t^2 - 5 = t^2$$
$$a_N = t$$

This allows us to express the acceleration vector as

$$\mathbf{a} = \sqrt{5}\mathbf{T} + t\mathbf{N}$$

From this value of a_N, we can easily obtain the curvature of the trajectory.

$$a_N = t = \frac{1}{\rho}v^2 = \kappa v^2 = \kappa(5t^2)$$
$$\kappa = \frac{1}{5t}$$

We learn from this question is that it pays to recognize whether it is easier to find the curvature κ from $\frac{1}{\rho}v^2 = a_N = \sqrt{\|\mathbf{a}\|^2 - a^2{}_T}$ or from $\left\|\frac{d\mathbf{T}}{ds}\right\|$. In this case, we pick the former because making the substitution $t = t(s)$ is tedious.