Differential Vector Calculus

Unit Binormal and Torsion

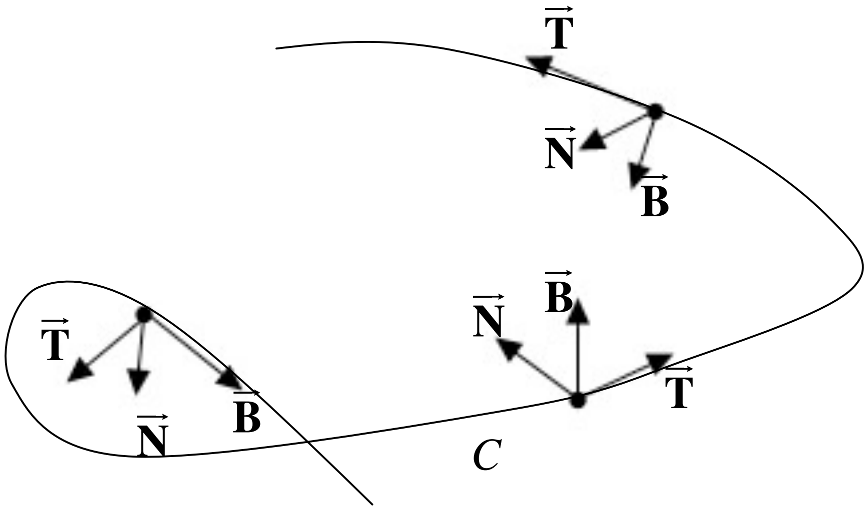
At any point on the curve C where $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ are defined, a third unit vector, the *unit binormal* $\vec{\mathbf{B}}$ is defined by the formula

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

With the three unit vectors, $\left\{\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}}\right\}$, they constitute a right-handed basis of mutually perpendicular unit vectors like the standard basis $\left\{\mathbf{i}, \mathbf{j}, \mathbf{k}\right\}$, the only difference being that it is right-handed.

To find out the directions of the vectors, take your hand and form a coordinate axis with your index, middle and thumb, all three being mutually perpendicular. Point your index finder in the direction of the first vector, your middle in that of the second, and your thumb will determine the direction of the third vector. Which direction the last vector will ultimately point depends on whether you are using your left or right hand. I should think it is obvious that when we specify the vectors in parenthesis, the order is important.

The basis of $\left\{\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}}\right\}$ is called the *frenet frame*.



Bearing in mind the directions, note that

$$\vec{\mathbf{N}} = \vec{\mathbf{B}} \times \vec{\mathbf{T}}$$

$$\vec{\mathbf{T}} = \vec{\mathbf{N}} \times \vec{\mathbf{B}}$$

We now wish to formulate some results from this frenet frame and the process we go about doing it is by differentiation. Just like how differentiating the position vector gives us the velocity vector, by differentiating these vectors, taking into account all such rules, we get a new set of equations.

From

$$\vec{\mathbf{B}} \bullet \vec{\mathbf{B}} = \left\|\vec{\mathbf{B}}\right\|^2 = 1$$

we differentiate with respect to *s*,

$$2\vec{\mathbf{B}} \bullet \frac{d\vec{\mathbf{B}}}{ds} = 0$$

And we conclude from here that $\frac{d\vec{\mathbf{B}}}{ds}$ is perpendicular to $\vec{\mathbf{B}}$. A result we will use later.

However, there's also another equation from which we can differentiate vector $\vec{\mathbf{B}}$, namely our definition $\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$. Doing so gives us

$$\begin{aligned} \frac{d\vec{\mathbf{B}}}{ds} &= \frac{d\vec{\mathbf{T}}}{ds} \times \vec{\mathbf{N}} + \vec{\mathbf{T}} \times \frac{d\vec{\mathbf{N}}}{ds} \\ &= \kappa\vec{\mathbf{N}} \times \vec{\mathbf{N}} + \vec{\mathbf{T}} \times \frac{d\vec{\mathbf{N}}}{ds} \\ &= \vec{\mathbf{T}} \times \frac{d\vec{\mathbf{N}}}{ds} \end{aligned}$$

which tells us that $\frac{d\vec{\mathbf{B}}}{ds}$ is also perpendicular to $\vec{\mathbf{T}}$. Now, here comes the tricky part. At first we seem to have new derivatives $\frac{d\vec{\mathbf{B}}}{ds}$ and $\frac{d\vec{\mathbf{N}}}{ds}$ which we have no idea about. But currently their direction is more important than their meaning. Imagine that vector $\vec{\mathbf{T}}$ is fixed in one direction. Our previous equation tells us that we can 'swivel' $\frac{d\vec{\mathbf{B}}}{ds}$ and $\frac{d\vec{\mathbf{N}}}{ds}$ round vector $\vec{\mathbf{T}}$ as long as they are mutually perpendicular. At this juncture, we make use of our previous equation that tells us $\frac{d\vec{\mathbf{B}}}{ds}$ is perpendicular to $\vec{\mathbf{B}}$, and as we know $\vec{\mathbf{T}}$ and $\vec{\mathbf{B}}$ are components from the frenet frame, the only possibility is that $\frac{d\vec{\mathbf{B}}}{ds}$ is parallel to $\vec{\mathbf{N}}$ which leads nicely to our definition of *torsion.*

Torsion given by the function $\tau(s)$ is define as

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau(s)\vec{\mathbf{N}}(s)$$

Torsion measures the degree of twisting that the curve exhibits near a point, that is to say, the extent to which the curve fails to be planar. It may be positive or negative depending on which direction the curve is twisting. A geometrical interpretation follows in the next lesson.