Differential Vector Calculus
Finding Unit Tangent and Unit Normal

We will continue from the previous lesson by looking at the same problem. To recall, we are given the position vector

$$\vec{\mathbf{F}}(t) = [\cos(t) + t\sin(t)]\mathbf{i} + [\sin(t) - t\cos(t)]\mathbf{j} + t^2\mathbf{k}$$

And found the velocity, acceleration vectors and curvature to be

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{F}}'(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{a} = \sqrt{5}\mathbf{T} + t\mathbf{N}$$

$$\kappa = \frac{1}{5t}$$

We will now explicitly find the unit tangent and unit normal vectors using clever manipulation of differentiating rules.

By chain rule, we can express the unit tangent as follows,

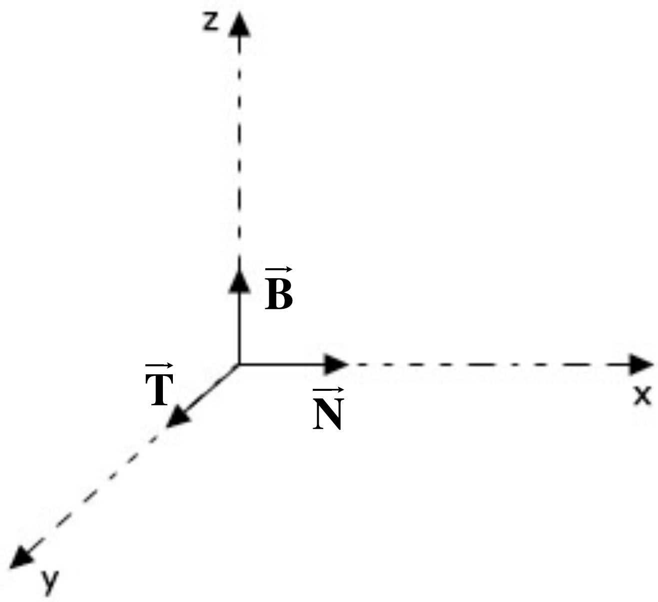
$$\begin{aligned}\vec{\mathbf{T}} &= \frac{d\vec{\mathbf{R}}}{ds} = \frac{dt}{ds}\frac{d\vec{\mathbf{R}}}{dt} = \frac{1}{v}\vec{\mathbf{v}} \\ &= \frac{1}{\sqrt{5}}\left[\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 2\mathbf{k}\right]\end{aligned}$$

To calculate the unit normal, we perform

$$\begin{aligned}\vec{\mathbf{N}} &= \rho\frac{d\vec{\mathbf{T}}}{ds} = \rho\frac{dt}{ds}\frac{d\vec{\mathbf{T}}}{dt} = \frac{1}{v}\rho\frac{d\vec{\mathbf{T}}}{dt} \\ &= \frac{5t}{\sqrt{5}t}\frac{1}{\sqrt{5}}\left[-\sin(t)\mathbf{i} + \cos(t)\mathbf{j}\right] \\ &= -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}\end{aligned}$$

Take note that $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ are orthogonal unit vectors.

We'll briefly introduce another vector called the *binormal* vector. The binormal vector is defined by $\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$ is a unit vector orthogonal to the plane of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. The vectors $\vec{\mathbf{N}}, \vec{\mathbf{T}}, \text{and}\, \vec{\mathbf{B}}$ form a right-handed coordinate system at each point on the curve and these three vectors is called the *frenet frame* which will be talked about in greater detail in the subsequent lesson.



For every point on the curve, $\vec{\mathbf{T}}, \vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$ exist and thus there's a frenet frame.