Differential Vector Calculus

Velocity and Acceleration

We shall now see how the terms velocity and acceleration, as many of you are familiar with in 2-dimensional kinematics, are applied to 3-dimensional vectors. We will have to use a few concepts of geometry of curves in this whole chapter to define some of the definitions.

Image a particle moving in 3D space. At time t, the particle is at the point $(x(t), y(t), z(t))$. The position vector would then be

$$\overrightarrow{\mathbf{F}}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

can be perceive as a movable arrow to the particle at time t. Much like previously thought, the functions $x = x(t), y = y(t), z = z(t)$ describes what we call the *trajectory* of the particle, giving us the associated curve in the space.

From here onwards, let's assume that x, y, and z are twice-differentiable functions of t. In any time interval $[t_1, t_2]$, the particle moves a distance along the trajectory given by

$$\int_{t_1}^{t_2} \left\| \overrightarrow{\mathbf{F}}'(t) \right\| dt$$

The *velocity* of the particle at time t is defined to be

$$\overrightarrow{\mathbf{v}}(t) = \overrightarrow{\mathbf{F}}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

The vector $\overrightarrow{\mathbf{v}}(t)$ is in the same direction of the tangent vector to the trajectory of the particle at time t. Loosely speaking, we can extract the magnitude of the velocity, otherwise known as the *speed*. It is a scalar quantity which we shall denote as v.

$$v(t) = \left\| \overrightarrow{\mathbf{v}}(t) \right\| = \overrightarrow{\mathbf{F}}'(t)$$

While this interpretation of the velocity vector and its magnitude seems correct, it is also consistent should we find the speed of the particle by finding the rate of change with respect to time of the distance traveled along the curve, as such.

$$\frac{d}{dt} \int_{t_0}^{t_1} \left\| \overrightarrow{\mathbf{F}}'(\xi) \right\| d\xi = \left\| \overrightarrow{\mathbf{F}}'(t) \right\| = \left\| \overrightarrow{\mathbf{v}}(t) \right\| = v(t)$$

Finally the acceleration of the particle is defined as the rate of change of velocity with respect to time, that is,

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

which I'll kindly remind you is a vector quantity.

Let's look are a simple example where we shall explicitly find the velocity and acceleration of a position vector.

Say, a particle is moving along a trajectory given by the parametric equations,

$$x = \sin(t), y = 2e^{-t}, z = t^2$$

From here, we get the position vector,

$$\vec{\mathbf{F}}(t) = \sin(t)\mathbf{i} + 2e^{-t}\mathbf{j} + t^2\mathbf{k}$$

The velocity vector is given by

$$\vec{\mathbf{v}}(t) = \cos(t)\mathbf{i} - 2e^{-t}\mathbf{j} + 2t\mathbf{k}$$

And the acceleration is

$$\vec{\mathbf{a}}(t) = -\sin(t)\mathbf{i} + 2e^{-t}\mathbf{j} + 2\mathbf{k}$$

and the speed is

$$v(t) = \left\|\vec{\mathbf{v}}(t)\right\| = \sqrt{\cos^2(t) + 4e^{-2t} + 4t^2}$$

Just a reminder, the symbols $\mathbf{v}(t)$ **and** $\vec{\mathbf{v}}(t)$ mean the same thing, the velocity vector v. While writing the v in bold is satisfactory to distinguish a vector from a scalar, sometime an arrow above the vector is used, as writing in bold on paper is inconvenient.