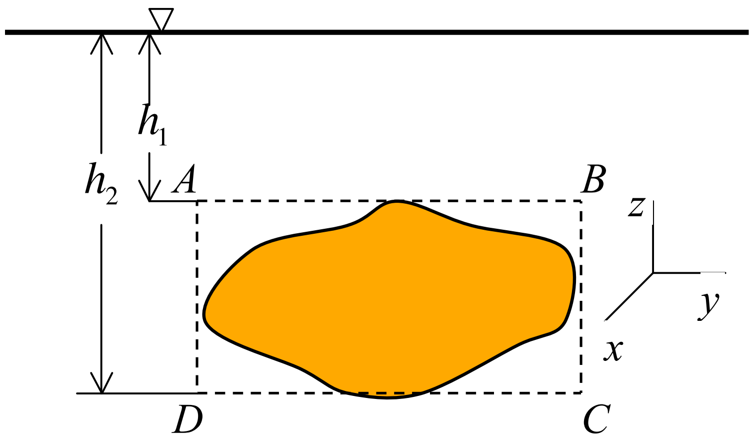
Fluid Mechanics

## Archimedes' Principle

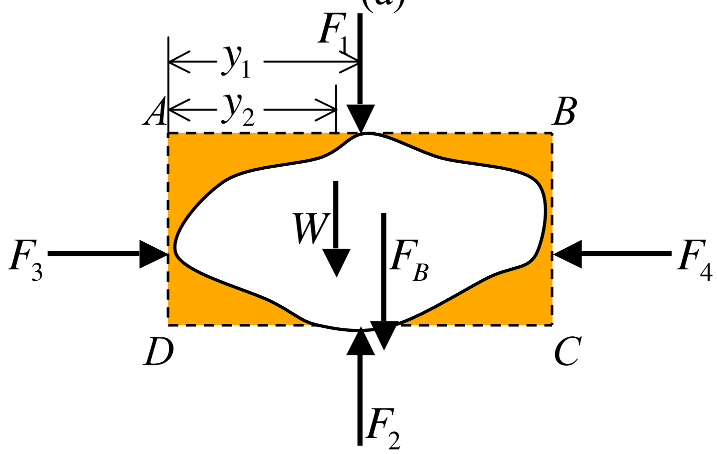
All of us have heard about Archimedes' Principle, that the force exerted on a body submerged in a liquid is equal to the weight of the displaced liquid. However, many of us may not know the proper derivation of the principle. This lesson is about using a mathematical method to derive the principle.

When a body is completely or partially submerged in a fluid, there is a resultant force acting on it called a *buoyant force*. The net upward vertical force results because pressure increases with depth and the pressure forces acting from below are larger than the pressure forces acting from above.

Consider a body having an arbitrary shape, with a volume $\forall$, that is immersed in a fluid as shown below as (a).




(a)




(b)

We enclosed the body in a parallelepiped and draw a free-body diagram of the *parallelepiped* with the body removed as shown in (b). Forces $F_1, F_2, F_3$

and $F_4$ are simply the forces exerted on the plane surfaces of the parallelepiped, ignoring forces in the *x* direction for simplicity. *W* is the weight of the shaded fluid volume, parallelepiped minus body, and $F_B$ is the force the body is exerting *on the fluid*.

The idea here is to find $F_B$ and by Newton's third law equate this force to that exerted on the body. The forces on the vertical surfaces, such as $F_3$ and $F_4$ are all equal and cancel so the equilibrium equation of interest is in the *z* direction.

$$F_B = F_2 - F_1 - W$$

Assuming specific weight of the fluid to be constant, then

$$F_2 - F_1 = \gamma(h_2 - h_1)A$$

where *A* is the horizontal area of the upper and lower surface of the parallelepiped. Combining both equations gives us

$$F_B = \gamma(h_2 - h_1)A - \left[\gamma(h_2 - h_1)A - \text{Ɏ}\right]$$

Observing that terms $h_1, h_2, A$ cancel out, we arrive at the expression for the buoyant force

$$F_B = \gamma \text{Ɏ}$$

where $\gamma$ is the specific weight of the fluid and Ɏ is the volume of the body. The direction of the buoyant force, which is the force of the fluid *on the body*, is opposite to that shown on the free-body diagram. Therefore, the buoyant force has a magnitude equal to $F_B$ but is directed vertically upwards. This result is commonly referred to as *Archimedes' principle* in honor of Archimedes, 287-212 B.C., way before Euler's time if you guys might be wondering.

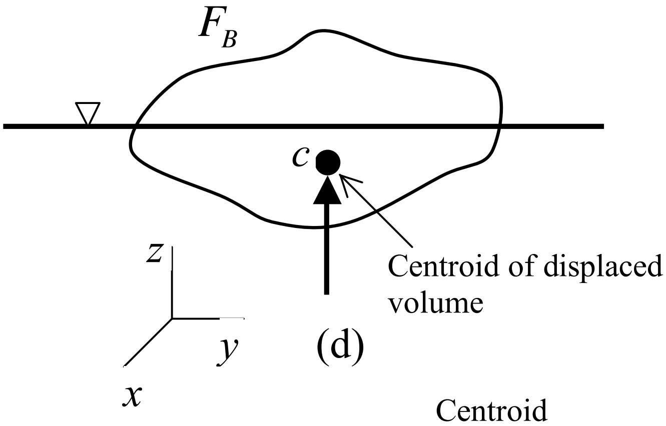
To determine the location of the line of action of the buoyant force, we sum up the moments of the forces shown in the free-body diagram with respect to some convenient axis. Summing moments about an axis perpendicular to the paper through point *D* gives us

$$F_B y_C = F_2 y_1 - F_1 y_1 - W y_2$$

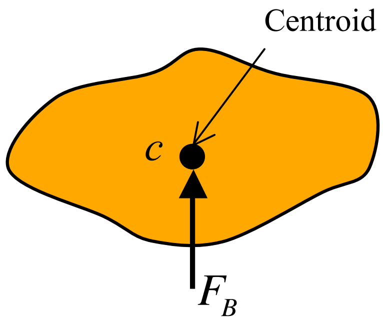
and on substituting for the various forces

$$\mathcal{V} y_C = \mathcal{V}_T y_1 - (\mathcal{V}_T - \mathcal{V}) y_2$$

where we define the new variable $\mathcal{V}_T$ as the total volume $(h_2 - h_1)A$. The right-hand side is the first moment of displaced volume $\mathcal{V}$ with respect to the *x*–*z* plane so that $y_C$ is equal to the *y* coordinate of the centroid of the volume $\mathcal{V}$. Similarly, the *x* coordinate of the buoyant force coincides with the *x* coordiate of the centroid*.* Thus, we conclude that *the buoyant force passes through the centroid of the displaced volume* as shown below (c). This point through which the buoyant force acts is called the *center of buoyancy*.




(d)




(c)

These same results apply to floating bodies which are only partially submerged, if the specific weight of the fluid above the liquid surface is very small compared with the liquid in which the body floats in (d). Usually, the fluid above the surface is air and so for practical purposes this condition is satisfied.