

Fluid Mechanics

Bernoulli's Equation after integration

In our previous lesson, we derive a form of Bernoulli's equation using Newton's second law applied to a fluid particle along a streamline. We now proceed with the integration, judiciously considering each term, to get the standard Bernoulli equation which we will use to solve problems.

The equation

$$-\gamma \sin \theta - \frac{\partial p}{\partial s} = \rho V \frac{\partial V}{\partial s}$$

can be rearranged and the integrated as follows. From the figure, we note that $\sin \theta = dz/ds$. We can also write $V dV/ds = \frac{1}{2} d(V^2)/ds$. Finally, given that along the streamline the change in pressure in the normal direction is zero, that is $dn = 0$, a small change in pressure in either normal or along the streamline given by $dp = (\partial p/\partial s)ds + (\partial p/\partial n)dn$ is now simply $dp = (\partial p/\partial s)ds$ or $\partial p/\partial s = dp/ds$. Substituting these variables into the previous equation gives us the following result applied along a streamline,

$$-\gamma \frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2} \rho \frac{d(V^2)}{ds}$$

which simplifies to

$$dp + \frac{1}{2} \rho d(V^2) + \gamma dz = 0$$

Remember that this equation is applied along the streamline. We divide throughout by ρ and proceed with the integration giving us

$$\int \frac{dp}{\rho} + \frac{1}{2} V^2 + gz = C$$

where C is a constant of integration to be determined by the condition at some point on the streamline.

Here is where we need to pay careful attention to the density term. Notice that I have isolated the density on the first term. In general, it is not possible to integrate the pressure term because the density need not be constant and therefore, cannot be removed from under the integral sign. To proceed further, we need to know how density varies with pressure which is not easy. We know that the perfect gas law relating density, pressure and temperature is $p = \rho RT$ where R is the gas constant. This still causes some complications because we then need to know the temperature variation.

So what do students do? As always, we assume density to be constant such as in an incompressible flow. Again this is a somewhat fair assumption for liquids and gases with low density but I suggest that the justification for this assumption be considered further.

Nonetheless, assuming that density remains constant, a good one for liquids and for gases if the speed is 'not too high', we can integrate the equation to give the following simple representation for steady, inviscid, incompressible flow.

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline}$$

This is the celebrated *Bernoulli equation* - a powerful tool in fluid mechanics. Daniel Bernoulli (1700-1782) published his *Hydrodynamics* in 1738 in which this famous equation appeared.

To correctly implement the equations, here are four basic assumptions used in its derivations:

1. Viscous effects are negligible.
2. The flow is assumed to be steady.
3. The flow is assumed to be incompressible.
4. The equation is applicable along a streamline.

A violation of these basic assumptions can lead to erroneous conclusions. Lastly, the constant of integration in the Bernoulli equation can be evaluated if sufficient information about the flow is known at one location along the streamline.