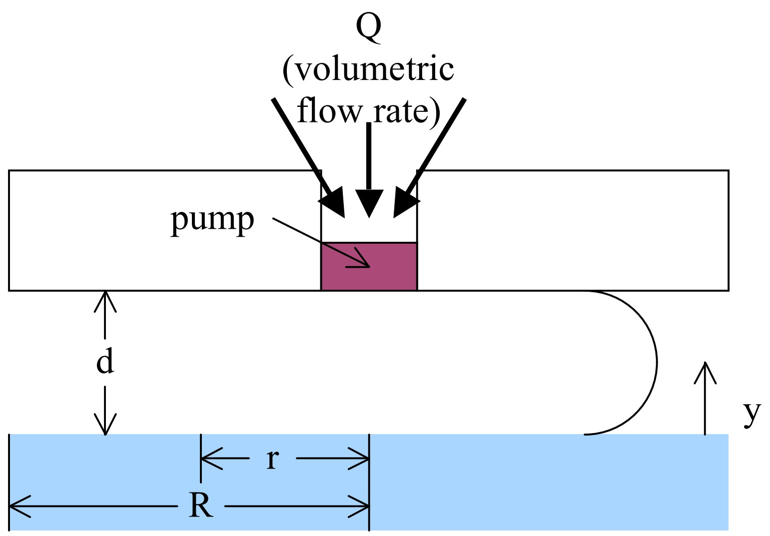
Fluid Mechanics
Analysis of the Hovercraft

Let's look at the concept of the Hovercraft, a classic fluid mechanics problem. There are a variety of ways to approach this, but we shall use Reynold's Transport Theorem. Later, we will go more in depth by looking at the pressure changes with Bernoulli's equation.

A hovercraft is made to hover above the water surface by delivering a cushion of air between the craft and the water surface. We model the hovercraft as a cylindrical object of diameter $2R$ and height H. Along the axis of cylinder is a uniform cylindrical shaft of diameter c (we'll use this value in the later lesson) where atmospheric air is sucked in at the rate of $Q(m^3s^{-1})$ by means of a powerful air pump placed near the bottom of the cylinder.



The sucked air is then pumped radically outwards in all directions thereby providing the cushion of air separating the cylinder from the water surface. As an approximation, we may assume that air is incompressible with a constant density, and that the water surface is taken as immobile. The radial velocity between the cylinder and the water surface is given by the parabolic velocity profile as

$$u(r,y) = ay(d-y)$$

noticing carefully that the velocity is dependent on *both* the radius r from the centre axis and y, the height from the water surface. d is the gap distance between the cylinder and the water surface. Take the dynamic viscosity of air as μ and the density of air as ρ.

Our first task is to determine *a(r)*. We will use Reynold's transport theorem to do so.

The theorem is given by

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\mathcal{V} + \int_{cs} \rho b \, \vec{\mathbf{V}} \bullet \hat{\mathbf{n}} \, dA$$

At first sight, you might be wondering why do we need such a big equation to solve this small problem, not to mention being intimidated by it. Well, Reynold's transport theorem can be used to solve a wide range of problems. The idea here is to start with this general equation and slowly set the terms to suit our problem. I believe that it is best that students get acquainted with this method as it makes them understand how the theorem works.

The first term to look at is the extensive property. Remember that *B* is the extensive property, the physical property we will be looking at, and *b* is the intensive property, the physical property per unit mass.

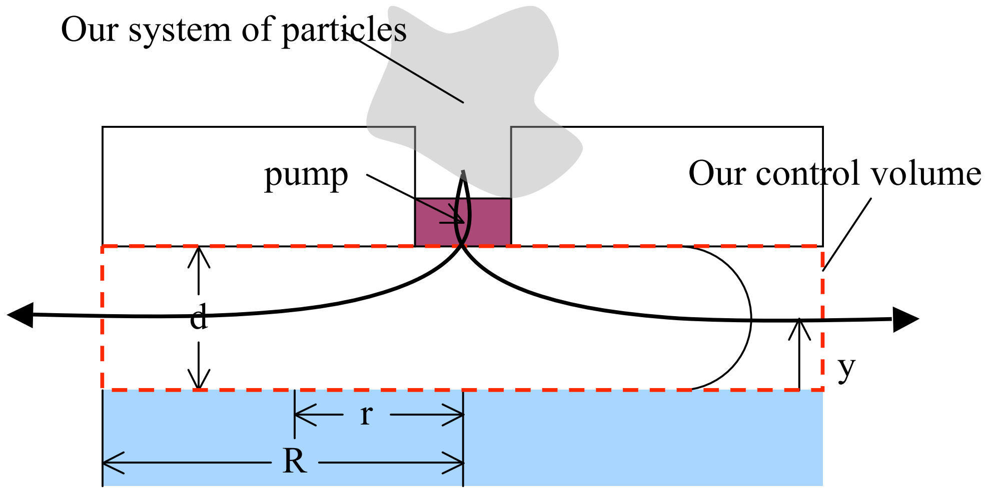
$$B = mb$$

So we ask what is the extensive property in our analysis. Well, it is the volume of air going through the and out of the hovercraft. To get the physical property of volume per unit mass, we divide mass by its density, and so $b = 1/\rho$ and substituting this into the equations yields

$$\frac{DV_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} d\mathcal{V} + \int_{cs} \vec{\mathbf{V}} \bullet \hat{\mathbf{n}} \, dA$$

Second, we look at the material derivative of the extensive property, which is now the material derivative of the volume of the system.

We now define our system of particles, which isn't too hard to do. Recognizing that a volume of air will start from outside the pump, go through the pump and then out radically, our system will be this volume of air particles as shown below.



Defining this to be our system, it should be easy to see that the material derivative is simply

$$\frac{DV_{sys}}{Dt} = 0$$

because as the air particles go through the pump, there is *no* change in its volume. Don't be shock to think the analysis is this easy. It actually is because our extensive property of volume makes it easy. If we choose our extensive property as, say velocity, things get much complicated.

The equation now becomes

$$\frac{\partial}{\partial t}\int_{cv} d\forall + \int_{cs} \vec{\mathbf{V}} \bullet \hat{\mathbf{n}}\, dA = 0$$

Thirdly, we have to define our control volume. In the above diagram, it is indicated by the dotted red line, or the volume where the air passes through and that demands our attention. So we ask ourselves, what is the rate of change of the volume of air particles in the control volume. Again, it is equal to zero because for a certain duration when air particles passes through, there is no change of the volume of air in the control volume. Another way to look at it is that for the hovercraft to stay afloat, air needs to be constantly fed in and released out.

By all these conditions, the equation is reduced to

$$\int_{cs} \vec{\mathbf{V}} \bullet \hat{\mathbf{n}}\, dA = 0$$

Recall what is this quantity. It is the net flow rate of volume through the control surface or rewriting as

$$\int_{cs} \vec{\mathbf{V}} \bullet \hat{\mathbf{n}}\, dA = \dot{V}_{out} - \dot{V}_{in}$$

giving us

$$\dot{V}_{out} - \dot{V}_{in} = 0$$

Now our calculations start. In all honestly, you could have written this calculation from the start but I emphasize again that this lesson is to see how to get to this equation *from* Reynold's Transport Theorem so that next time when the extensive property is something other than volume, you are prepared in handling it.

From here on it's using methods of calculus which many of you should be familiar with.

$$\dot{V}_{in} = \dot{V}_{out}$$
$$Q = \int_{cs} u\, dA$$

Volumetric flow rate into the control surface is given by *Q*. For the volumetric flow rate out of the control surface, we multiply the velocity *u* at a small area *dA* bearing in mind that *u* is not constant throughout the control surface.

$$\begin{aligned} Q &= \int_{cs} u\, dA \\ &= \int_0^d ay(d-y)2\pi r\, dy \\ &= \frac{\pi r a d^3}{3} \end{aligned}$$

implying that

$$a = \frac{3Q}{\pi r d^3}$$

as expected.

So, using Reynold's Transport Theorem, applying the relevant equations for the extensive property and reducing the material derivative and rate of change to zero, we are able to find the velocity function.

$$u(r, y) = \frac{3Q}{\pi r d^3} y(d - y)$$

Again, I stress that canceling the certain quantities to zero is acceptable here because the extensive property being volume, is simple to analyze. In the subsequent lessons, we take the idea of the hovercraft a step further and find the force required to hold it up using Bernoulli's equation.