Fluid Mechanics

Pressure Variation for Fluid at Rest – Incompressible

Now that we have our basic equation for the pressure gradient, that is $-\nabla p - \gamma \hat{\mathbf{k}} = \rho \vec{\mathbf{a}}$, we shall apply it to fluids with certain conditions to find the pressure gradient for that particular fluid. We start with fluids at rest and then later consider the case that they are incompressible.

For a fluid at rest, $\vec{\mathbf{a}} = 0$, the basic equation reduces to

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \vec{\mathbf{a}}$$

or in component form

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma$$

Pay particular attention that from the differential equations the pressure does not depend on *x* or *y*. This means that as we move from point to point in a horizontal plane, or any place parallel to the *x*–*y* plane for that matter, the pressure does not change. Our focus would be that *p* depends only on *z*, rewriting the last equation as

$$\frac{dp}{dz} = -\gamma$$

This is the fundamental equation for fluid at rest and will be used to determine how pressure changes with eleveantion. The negative sign should be no surprise as many of us expect as we go up from the base of the fluid, the pressure decreases. Lastly, there is no requirement that γ is constant. Remember that $\gamma = \rho g$. At any event, the density ρ need not be a constant throughout the liquid, though it is assumed to be constant for particular fluids, which we shall now see.

For most engineering applications the variation in *g* is negligible, so our main concern is with the possible variation in the fluid density. This value is usually negligible, over large distances, so the assumption of constant specific weight when dealing with liquids is a good one. We thus can direction integrate our dp/dz equation.

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz$$

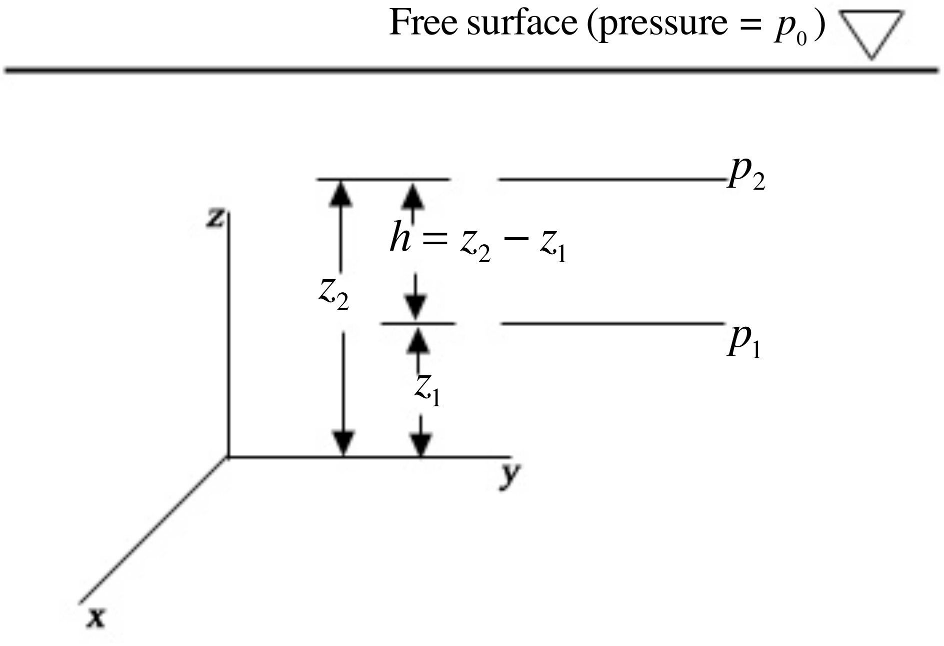
to get

$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

or rearranging

$$p_1 - p_2 = \gamma(z_2 - z_1)$$

where p_1 and p_2 are pressures at the vertical elevations z_1 and z_2, as illustrated below.



We introduce *h* as the difference in height and write

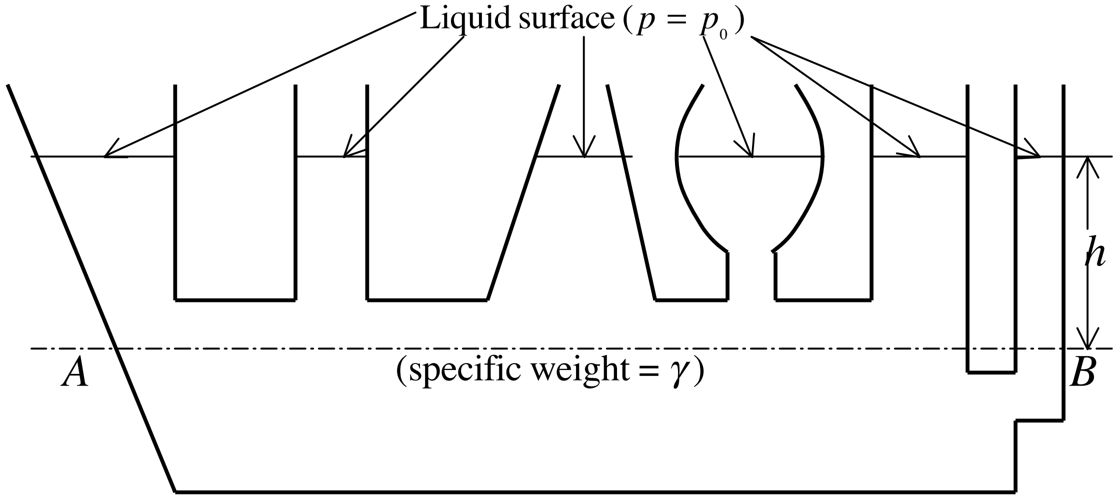
$$p_1 - p_2 = \gamma h$$
$$p_1 = \gamma h + p_2$$

where *h* is the distance $z_2 - z_1$, which is the depth of fluid measures downward from the location of p_2. This type of pressure variation is commonly called a *hydrostatic distribution* where for an incompressible fluid at rest, the pressure varies linearly wiht depth. The pressure must increase to “hold up” the fluid above it.

When working with liquids, there is often a free surface as illustrated in our previous figure, and it is convenient to use this surface is a reference plane. The reference pressure p_0 would correspond to the pressure acting on the free surface. By letting $p_2 = p_0$, it follows that the pressure *p* at any depth *h* below the free surface is given by

$$p = \gamma h + p_0$$

I would like to emphasize again that the pressure in an incompressible fluid at rest depends on the depth of the fluid relative to some reference and it is *not* influenced by the *size* or *shape* of the take holding the fluid. Hence, if we have the container below,



the pressure is the same at all points along the line *AB* even though the container may have a very irregular shape. The actual pressure along the line *AB* depends only on the depth, *h*, the surface pressure p_0, and the specific weight, γ, of the liquid in the container.