Fourier Analysis

Convergence theorem of a Fourier Series

We go straight to defining the convergence theorem of a Fourier series and soon after look at some examples of its application.

Let $f(x)$ be piecewise continuous on [–*L*, *L*].

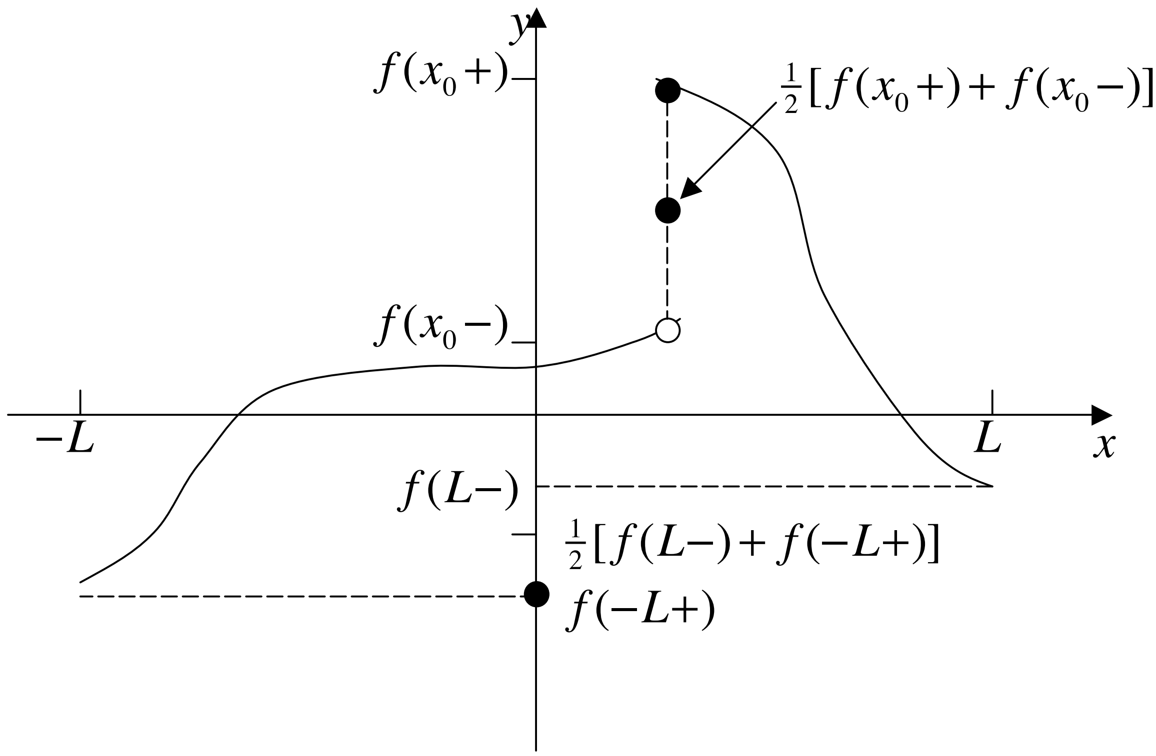
1. If $-L < x_0 < L$ and both left and right derivatives of $f(x)$ exist at x_0, the Fourier series of $f(x)$ on [–*L*, *L*] converges at x_0 to

$$\tfrac{1}{2}[f(x_0+)+f(x_0-)]$$

2. If both $f'_R(-L)$ and $f'_L(L)$ exist, the Fourier series converges at both *L* and –*L* to

$$\tfrac{1}{2}[f(-L+)+f(L-)]$$

The figure below is an illustration of the theorem. Let us talk a little more in depth about it.



At each point between $-L$ and L where the right and left derivatives exist, the Fourier series converges to the average of the left and right limits of $f(x)$. This fact is illustrated above. In particular *if* $f(x)$ *is continuous at* x_0 *for the whole domain of x*, the left and right limits both equal $f(x_0)$ and the series converges at x_0 to

$$\tfrac{1}{2}[f(x_0+)+f(x_0-)] \text{ or } f(x_0)$$

While you may be quick to think that since all functions are usually continuous in the domain without any jump discontinuity, pay close attention to the domain for which the theorem applies namely $-L < x_0 < L$, a strict inequality. Convergence for the endpoints is tested using the second part of the theorem.

At *both* $-L$ and L, the Fourier series converges to the average of the right limit at $-L$ and the left limit at L, assuming that the respective derivatives exist at both points. It should not be surprising that the Fourier series converges to the same value at L and at $-L$. Upon letting $x = L$ in the Fourier series of $f(x)$, we get

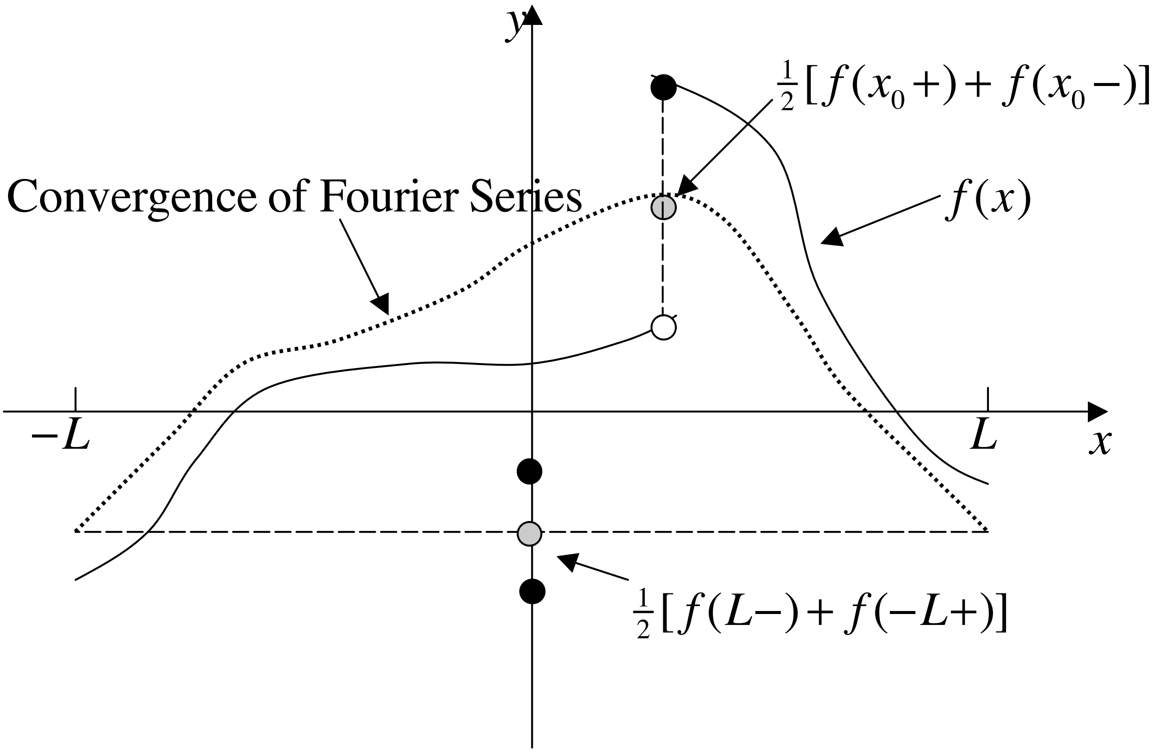
$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi)$$

because $\sin(n\pi) = 0$. If we let $x = -L$ in the Fourier series, we get exactly the same series because $\sin(-n\pi) = 0$ and $\cos(-n\pi) = \cos(n\pi)$.

In terms of the graph, the Fourier series converges at $-L$ and at L to the point midway between the ends of the graph at those points.

In light of the above theorem, we can often tell what the Fourier series of $f(x)$ converges to just by looking at the graph of $f(x)$. Really, it is a matter of taking the averages of the value of the function at the jump discontinuous and the average of the function at the end points.

Using our previous graph of an example, I think this is how the Fourier series will converge.



Last point to note: while it may be obvious to some, notice that the information of the convergence of the Fourier series comes from the function $f(x)$ and NOT the Fourier series itself. Yes, there may be a few convergence test we can use on the Fourier series itself, but the above theorem is applied to the function. In other words, the function $f(x)$ tells us whether its Fourier series converges to itself.