Fourier Analysis

Definition of the Fourier Series

Our journey in learning Fourier analysis begins with question of what exactly is the Fourier series. There many kinds of Fourier series, with different conditions, we start by looking at the most basic one.

Suppose *f* is integrable on $[-\pi,\pi]$. We want to entertain the possibility of choosing constants $a_0, a_1, \ldots, b_1, b_2, \ldots$ so that

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

Under fairly general conditions on *f*, we can write such an equation, except possibly at a finitely many points of $[-\pi,\pi]$. Our job here is to find a method to obtain a_n's and b_n's. The key is in a few innocent looking lemmas of trigonometry functions.

The proof of the two lemmas which follows is rather lengthy. To go straight to the derivation of the series, click here.

Lemma 1

1. If *n* and *m* are distinct nonnegative integers,

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx = \int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx = 0$$

2. For any positive integers *m* and *n*,

$$\int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx = 0$$

Both of these results can be obtained via integration. Using the following formulas, we get

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx=\left[\frac{\sin[(n-m)x]}{2(n-m)}+\frac{\sin[(n+m)x]}{2(n+m)}\right]_{-\pi}^{\pi}=0$$

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx=\left[\frac{\sin[(n-m)x]}{2(n-m)}-\frac{\sin[(n+m)x]}{2(n+m)}\right]_{-\pi}^{\pi}=0$$

If m=0 and $n\neq$0,

$$\int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx=\int_{-\pi}^{\pi}\cos(nx)dx=\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi}=0$$

If n and m are positive integers and $m\neq n$,

$$\int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx=\left[-\frac{\cos[(n-m)x]}{2(n-m)}-\frac{\cos[(n+m)x]}{2(n+m)}\right]_{-\pi}^{\pi}=0$$

because cos(A)=cos(-A) for any A. If n and m are positive integers and $m=n$,

$$\int_{-\pi}^{\pi}\cos(nx)\sin(nx)dx=\left[\frac{1}{2n}\sin^2(nx)\right]_{-\pi}^{\pi}=0$$

Lastly, $\int_{-\pi}^{\pi}\cos(mx)dx=\int_{-\pi}^{\pi}\sin(mx)dx=0,\text{for m}=0,1,2,\ldots$

Briefly commenting, getting 0 for these integration results should be no surprise because we are integrating cosine or sine function from $-\pi$ to π, limits which when substituted usually result in a 0. These formulas are called *orthogonality relationships*, and the functions cos(nx) for n = 0, 1, 2, …, and sin(nx) for n = 0, 1, 2, … are said to be *orthogonal* on $[-\pi,\pi]$.

Lemma 2
For any positive integer n,

$$\int_{-\pi}^{\pi}\cos^2(nx)dx=\int_{-\pi}^{\pi}\sin^2(nx)dx=\pi$$

As with the previous lemma, the proof is by routine integration.

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[1+\cos(2nx)\right] dx = \left[\frac{1}{2}x + \frac{1}{2n}\sin(2nx)\right]_{-\pi}^{\pi} = \pi$$

Similarly,

$$\int_{-\pi}^{\pi} \sin^2(nx)dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[1-\cos(2nx)\right] dx = \pi$$

With these lemmas in place, we can now return to the question of how *f* can be written as a series of sines and cosines together with a constant term

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

for $-\pi \le x \le \pi$, calling it the Fourier series equation. We use an informal argument suggesting how we should choose the a_n's and b_n's. The idea here is to use the previous results to filter out terms by integrating and then solving to find the constant terms.

Integrate both sides of the equation from $-\pi$ to π, and assume for the moment that we can interchange the summation and the integral. We get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)dx + b_n\int_{-\pi}^{\pi}\sin(nx)dx\right] \\ &= 2\pi a_0\end{aligned}$$

because all of the integrals in the summation are zero. Solving this equation yields

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

Now let *k* be any positive integer. We will 'filter' out a_k. Multiply our Fourier series equation this time by cos(*kx*) to get

$$f(x)\cos(kx) = a_0\cos(kx) + \sum_{n=1}^{\infty}\left[a_n\cos(nx)\cos(kx) + b_n\sin(nx)\cos(kx)\right]$$

Doing the same by integrating from $-\pi$ to π, and again interchanging the integral and the summation, we get

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = a_0\int_{-\pi}^{\pi}\cos(kx)dx$$
$$+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\cos(kx)dx + b_n\int_{-\pi}^{\pi}\sin(nx)\cos(kx)dx\right]$$

By our first lemma, all of the integrals on the right are zero except the one involving cos(*nx*)cos(*kx*) when *n* = *k*. If you have a hard time looking that this, just imaging running the variable 1 to ∞ until you reach somewhere in the sequence of say 5,6,7,…,*k*−1,*k*,*k+1*,…. The second integral in the summation will always be 0. The first integral will be zero except when the two cosine terms are equal, that is cos(*kx*)cos(*kx*). The last equation therefore collapses to just

$$\int_{-\pi}^{\pi} f(x)\cos(kx)dx = a_k\int_{-\pi}^{\pi}\cos(kx)\cos(kx)dx = a_k\pi$$

by our second lemma. Solving this equation for a_k to get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{-} f(x)\cos(kx)dx$$

for *k* = 1, 2, 3, …

To solve for b_k by using a similar argument, we multiply the Fourier series equation this time by sin(*kx*) and again use the orthogonality relationships to deduced that all integrals collapses except for $\int_{-\pi}^{\pi}\sin(nx)\sin(kx)dx$ when *n* = *k* in the summation. It reduces to

$$\int_{-\pi}^{\pi} f(x)\sin(kx)dx = b_k\int_{-\pi}^{\pi}\sin^2(kx)dx = b_k\pi$$

which we can conclude that

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

for *k* = 1, 2, 3, …

I would like to point out that this derivation is somewhat flawed by interchanging the summation $\sum_{k=1}^{\infty}$ with the integral $\int_{-\pi}^{\pi} dx$. Nonetheless, till we prove that we can in fact do that, we shall take the mathematics as it is. We wrap up by a formal definition of the Fourier Series.

Let $f(x)$ be integrable on $[-\pi,\ \pi]$.
1. The *Fourier coefficients* of $f(x)$ on $[-\pi,\ \pi]$ are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \qquad \text{for } n = 1,2,3,\ldots \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \qquad \text{for } n = 1,2,3,\ldots
\end{aligned}$$

2. The *Fourier series of* $f(x)$ on $[-\pi,\ \pi]$ is the series

$$a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

in which the coefficients are the Fourier coefficient of $f(x)$ on $[-\pi,\ \pi]$.

That was a mouthful of theory for you to digest. Again, while this five page mathematical explanation may seem of little relevance in terms of application to problems, I strongly advise a thorough reading to grasp the genius behind the work, and some work it has been.