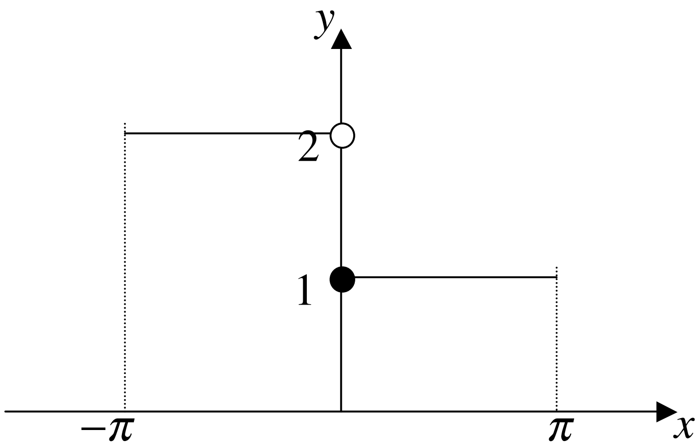
Fourier Analysis

Fourier Series of a ‘Broken’ function

With our newly acquired knowledge of the definition of the Fourier Series, let’s put what we have learnt in theory to good use. In Fourier analysis, we usually find Fourier series either graphs or functions explicitly defined.

Our objective is to find the Fourier Series of the function *f*(x) given by the graph below:



Finding the Fourier Series can be classified into four steps:
1. Get the function from the graph (if not explicitly defined).
2. Find the Fourier Coefficients (integration by parts usually needed).
3. Write the Fourier Series.
4. Consider ‘even’ and ‘odd’ cases.

Since we are not Fourier analysis experts, I shall systematically go through the four steps.

1. Get the function from the graph.

We may omit this step if the function is explicitly given for us. This problem gives us only a graph of the function and so our task is to define a function, which shouldn’t be a problem. It is

$$f(x) = \begin{cases} 2, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

I shall point out a few things. One, be careful of the inequality signs for the domain, in this case, the unfilled dot at (0,2) means *y*=0 for *x* strictly less than 0. Two, such functions which we define from the graph are usually ‘simple’ in that the *y* value is a constant term. For other problems where we are given the equations of the graphs, our task is reduced to specifying the domain of *x* in which the function follows that part of the graph. Third, I like to call these functions as ‘broken’ functions because more often than not, we end up splitting the domain of *x*.

2. Find the Fourier Coefficients (integration by parts usually needed).
Once we have the function, we immediately apply the relevant formulas to find a_0, a_n and b_n. An early warning is that integration by parts is usually needed, though not for this example, as we are integrating products of *x* and sin(*nx*) or cos(*nx*).

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 2dx + \int_{0}^{\pi} 1dx\right)$$

Notice here that we have to split the integrals because for the two domains of *x*, *f(x)* is different. Easily integrating we get,

$$a_0 = \frac{1}{2\pi}\left(2\pi + \pi\right) = \frac{3}{2}$$

Moving swiftly along, we apply the same formula to find a_n,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\cos(nx)dx + \int_{0}^{\pi} \cos(nx)dx\right)$$

First, be aware of the coefficients before the integral sign. For a_n and b_n, it is $1/\pi$ but for a_0, it is $1/2\pi$. Let’s clear up this potentially concealed mistake before you start making further errors. Again, we have to split the integrals, integrating the respective function in the separate domain of $[-\pi, 0]$ and $[0, \pi]$. This some situations and with a perfect use of algebra, you are at liberty to combine the integrals within certain limits to simplify the working. That technique is explored in subsequent examples.

$$a_n = \frac{1}{\pi}\left\{2\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{0} + \left[\frac{1}{n}\sin(nx)\right]_{0}^{\pi}\right\} = \frac{1}{\pi}\{0\} = 0$$

using the common identity $\sin(n\pi) = 0$ to evaluate this integral. We get a zero for a_n. This is NO cause for concern. It isn’t rare to get zero for either a_n or b_n in Fourier series problems. This is attributed to integrating of sine and cosine functions in the limits $[-\pi, \pi]$. Now, we do the same for b_n – apply the formula, split the integral and integrate accordingly.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 2\sin(nx)dx + \int_{0}^{\pi}\sin(nx)dx\right)$$
$$= \frac{1}{\pi}\left(\frac{-2}{n}\big(1-\cos(n\pi)\big) + \frac{-1}{n}\big(\cos(n\pi)-1\big)\right)$$

At this stage, we recall another common identity which is $\cos(n\pi) = (-1)^n$, to give us

$$b_n = \frac{1}{\pi}\left(\frac{-2}{n}\big(1-(-1)^n\big) - \frac{1}{n}\big((-1)^n - 1\big)\right)$$
$$= \frac{1}{n\pi}\big(-1+(-1)^n\big)$$

3. Write the Fourier Series.
With our coefficients a_0, a_n and b_n, we substitute them into the Fourier series equation

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

giving us the Fourier series of *f*(x) as

$$\frac{3}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\big(-1+(-1)^n\big)\sin(nx)$$

Remember that after substituting the coefficients a_n and b_n, only b_n in this case, we have to multiply it with cos(*nx*) and sin(*nx*) respectively for the summation. This is different from multiplying cos(*nx*) and sin(*nx*) with *f*(*x*) when we did the integration in step 2. Just be aware of that.

4. Consider 'even' and 'odd' cases.
The Fourier series we wrote out is perfectly correct. However, there is a way to simplify the otherwise messy looking equation. Notice that *n* can

take either even or odd values and that the $\left(-1+(-1)^n\right)$ term in the summation is equal to 0 for even values of *n*. This motivates us to neaten the equation by considering even and odd cases for the summation.

When *n* is even, that is *n*=2*m*, we have

$$\sum_{m=1}^{\infty}\frac{1}{2m\pi}\left(-1+(-1)^{2m}\right)\sin(2mx)=\sum_{m=1}^{\infty}\frac{1}{2m\pi}\left(-1+1\right)\sin(2mx)=0$$

When *n* is odd, that is *n*=2m−1, we have

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)\pi}\left(-1+(-1)^{2m-1}\right)\sin(2m-1)x=$$

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)\pi}\left(-1+-1\right)\sin(2m-1)x=\frac{-2}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2m-1)x}{(2m-1)}$$

Don't worry too much why it is *n*=2m−1 and not *n*=2m+1. A simple explanation is because we start with *m*=1 which corresponds with *n*=1 when *n*=2m−1. Lastly, we swap the dummy variable from *m* back to *n* for consistency sake. This is perfectly fine because be it *m* and *n*, it hold no significance other than counting from 1 to ∞. So the Fourier series finally becomes.

$$\frac{3}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

Without revealing too much of the convergence theorem, we are careful to not equate the function *f*(*x*) with the Fourier series. Just keep this as a thought. It will be investigated further in the lessons to come. Right now we are concern with 'finding the Fourier series of *f*(*x*)' and not '*f*(x)=…the Fourier series…'