Fourier Analysis

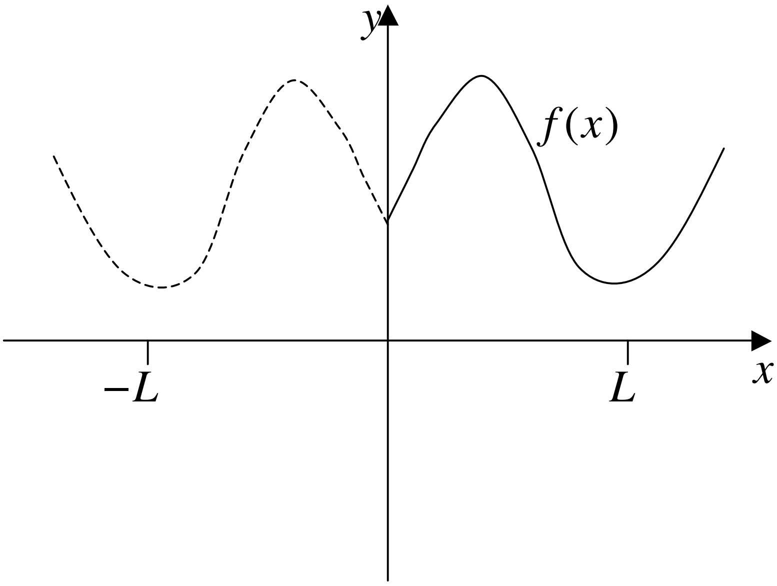
Fourier Cosine Series

In this lesson, we take a step further and explore the possibilities of writing a Fourier series of *f* on [0,*L*] containing either just sine terms or just cosine terms, whicever we chooses. Such series are called *half-range expansions*. The key lies in a previous lemma.

Suppose that *f* is integrable on [0,*L*] and we want to expand *f* in a Fourier series on [0,*L*] containing just cosine terms. The idea is to extend *f* to a new function *g* defined on [-*L*,*L*] in such a way that *g* is an even function. The Fourier series of *g* on [-*L*,*L*] contains only cosine terms. Since *f* and *g* agree on [0,*L*], this gives a Fourier cosine series of *f* on [0,*L*]. To do this, define

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L \\ f(-x) & \text{for } -L \le x \le 0 \end{cases}$$

A typical graph of *g* is obtained by folding the graph of *g* over the *y*-axis as shown below.



Since *g* is an even function, its Fourier series on [-*L*,*L*] is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

in which

$$a_0 = \frac{1}{L}\int_0^L g(x)dx \text{ and } a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Since *g*(x)=*f*(x) for $0 \le x \le L$, we may think of the series as a Fourier cosine series of *f*(x) on [0,*L*]. Further, the coefficeints can be written in terms of *f*. This leads to a formal defintion of the Fourier cosine series.

If *f* is integrable on [0,*L*], the *Fourier cosine series* of *f* on [0,*L*], is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \text{ and } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

We remark that the function *g* we introduced had the sole purpose of suggesting how to define a pure cosine series on [0,*L*]. In computing a Fourier cosine series, we never mention *g* but simply calculate the coefficeint directly from the definition.

As with all Fourier series, there is a convergence test for the Fourier cosine series.

Let *f* be piecewise continuous on [0,*L*].
1. If $0 < x < L$, and *f* has both right and left derivatives at *x*, then at *x* the Fourier cosine series of *f* on [0,*L*] converges to

$$\tfrac{1}{2}\left[f(x+) + f(x-)\right],$$

the average of the left and right limits of *f* at *x*. In particular, if *f* is also continuous at *x,* the series converges to *f*(*x*).
2. If $f'_R(0)$ exists, the cosine series converges at zero to $f(0+)$.
3. If $f'_L(0)$ exists, the cosine series converges at *L* to $f(L-)$.

Conclusion (1) should be no surprise. To understand conclusion (2), consider the convergence at zero for the Fourier series of *g* on [−*L*,*L*]. At zero, this series converges to $\frac{1}{2}\left[g(0+)+g(0-)\right]$. But

$$g(0+)=\lim_{h\to 0^+} g(0+h)=\lim_{h\to 0^+} g(h)=\lim_{h\to 0^+} f(h)=f(0+)$$

and

$$g(0-)=\lim_{h\to 0^+} g(0-h)=\lim_{h\to 0^+} g(-h)=\lim_{h\to 0^+} f(h)=f(0+)$$

and so

$$\tfrac{1}{2}\left[g(0+)+g(0-)\right]=\tfrac{1}{2}\left[f(0+)+f(0-)\right]=f(0+),$$

and the cosine series of *f* converges to $f(0+)$ at zero. A similar argument establishes convergence of the series to *f*(*L*−) at *L*.