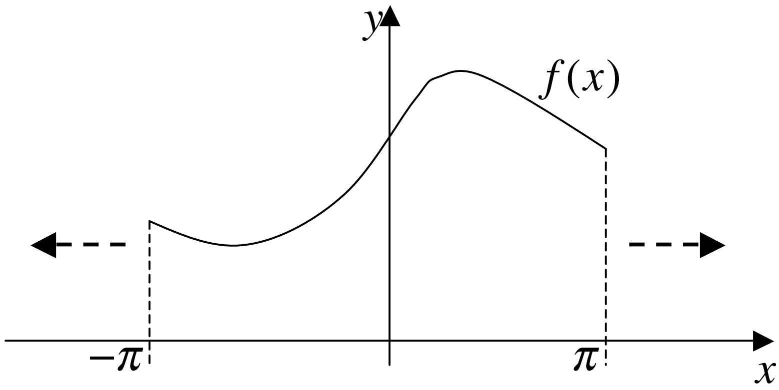
Fourier Analysis

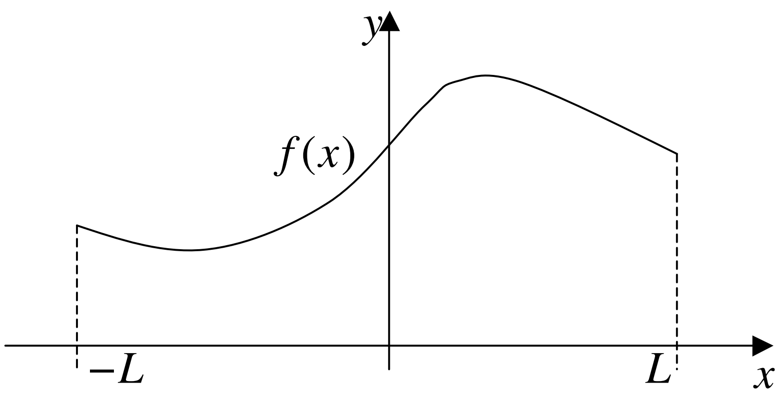
Fourier Series of a function on [-*L*, *L*]

Our first Fourier series definition started with the function integrable from $[-\pi, \pi]$. However, most functions are usually defined for an arbituary domain, say from -*L* to *L*. With a small modification to the original definition, we can get the Fourier series of such functions.

Finding the Fourier series of the function *f*(*x*) shown graphically below shouldn't be a problem. Since the function is defined on $[-\pi, \pi]$, we can easily apply the formulas.



Unless all the functions in the world are defined on $[-\pi, \pi]$, we will be satisfied with ourselves. Sadly, this is not the case and rightly so. How do we find the Fourier series of a function on [-*L*, *L*] graphically shown below.



The method is rather simple. We will use integration by substitution with a change in variable to alter the original definition to suit this problem.

We first need to relate two independent variables which define the *x* axis of the two graphs. Let them be *t* and *x*, and that

$$t = \frac{\pi x}{L} \text{ and so } x = \frac{\pi L}{t}$$

Notice here the respective domains for both independent variables. *x* is defined on [-*L*, *L*] and *t* is defined on $[-\pi, \pi]$. Next, we write a function in terms of *t* which translates the function *f*(*x*) such that

$$f(x) = g(t)$$

Now, we can use the previous definitions to find the Fourier coefficients of function *g*(*t*) as the domain of *t* is $[-\pi, \pi]$, and from there, from the respective Fourier coefficients of *f*(*x*).

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt = \frac{1}{2\pi}\int_{-L}^{L} g(t)\frac{\pi}{L}dx$$
$$= \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

We have employed the method of integration by substitution using the relationship $dt = (\pi/L)dx$. Doing the same for the other coefficients,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt = \frac{1}{\pi}\int_{-L}^{L} g(t)\cos(nt)\frac{\pi}{L}dx$$
$$= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)dt = \frac{1}{\pi}\int_{-L}^{L} g(t)\sin(nt)\frac{\pi}{L}dx$$
$$= \frac{1}{2L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

That wasn't too hard. So our formal definition of the Fourier series and coefficients on [-*L*, *L*] is as follows.

Let $f(x)$ be integrable on $[-L, L]$.

1. The *Fourier Coefficients* of $f(x)$ on $[-L, L]$ are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ and}$$

$$b_n = \frac{1}{2L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

for n = 1, 2, 3, ...

2. The *Fourier Series* of $f(x)$ on $[-L, L]$ is

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

in which the numbers $a_0, a_1, \ldots, b_1, \ldots$ are the Fourier coefficients of $f(x)$ on $[-L, L]$.

For me, the most noticeable change is the variable inside the trigonometry functions. Once you can get used to the change from *nx* to $n\pi x / L$, you should be fine. Now we can find the Fourier Series for another set of functions.