Fourier Analysis

Half Range Expansion of f(x)=e^2x

Now that we know how to find the half-range expansion of a function by writing the Fourier series on [0,*L*] in terms of either cosine or sine terms, we shall look at a typical function as an example.

The objective here is to write both the half-range expansion in cosine and sine terms of the function $f(x) = e^{2x}$ and compare the two. We first write the Fourier cosine series of *f* on [0,1].

The coefficients are

$$a_0 = \tfrac{1}{1}\int_0^1 e^{2x}dx = \tfrac{1}{2}(e^2 - 1)$$

and

$$a_n = \frac{2}{1}\int_0^1 e^{2x}\cos(n\pi x)dx = \frac{4}{4+n^2\pi^2}\left[e^2\cos(n\pi) - 1\right]$$

The Fourier cosine series of $f(x) = e^{2x}$ on [0,1]. is

$$\tfrac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty}\frac{4}{4+n^2\pi^2}\left[e^2\cos(n\pi) - 1\right]\cos(n\pi x)$$

or, since $\cos(n\pi) = (-1)^n$,

$$\tfrac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty}\frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]\cos(n\pi x)$$

We proceed by applying the convergence theorem to find the domain in which this Fourier cosine series converges to the function $f(x) = e^{2x}$. By inspection, we know that for $0 < x < L$, *f*(*x*) is continuous at *x* an so the series converges to *f*(x).

At x=0 where $f'_R(0)$ exist, the Fourier cosine series converges to $f(0+) = e^{2\cdot 0} = 1$ and at x=1 where $f'_L(0)$ exist, the Fourier cosine series converges to $f(1-) = e^{2\cdot 1-} = e^2$

Let's move on to find the Fourier sine series of the same function on the same domain. Immediately applying the formula to find the coefficients,

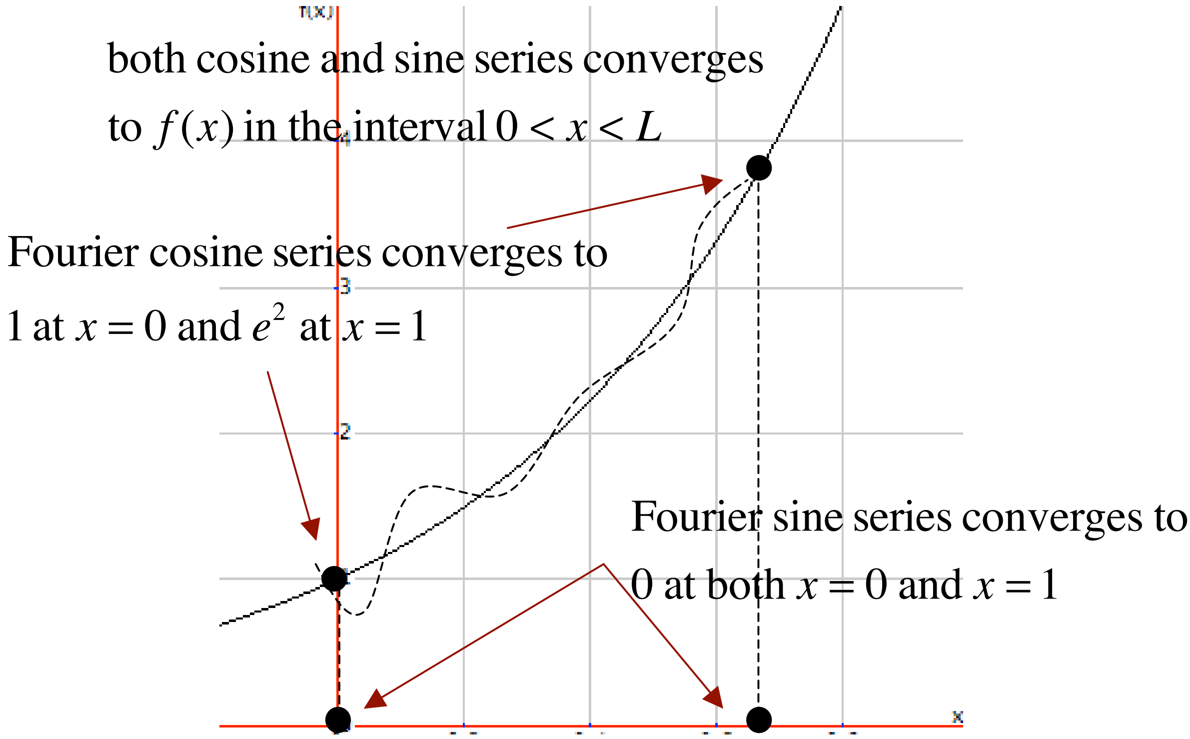
$$b_n = \frac{2}{1}\int e^{2x}\sin(n\pi x)dx = \frac{2n\pi\left[1 - e^2\cos(n\pi)\right]}{4 + n^2\pi^2}$$
$$= \frac{2n\pi\left[1 - e^2(-1)^n\right]}{4 + n^2\pi^2}$$

The sine series of $f(x) = e^{2x}$ on [0,1] is

$$\sum_{n=1}^{\infty}\frac{2n\pi\left[1 - e^2(-1)^n\right]}{4 + n^2\pi^2}\sin(n\pi x)$$

We first notice that both the cosine and sine series are rather different yet they are the expansions of the *same* function. Let's see what results we get when we applying the convergence theorem for the sine series.

Again we know that for $0 < x < L$, $f(x)$ is continuous at x an so the series converges to f(x). When looking at the endpoints, the theorem immediately tells us that at both 0 and 1, the sine series of f converges to zero, a very different result compared with the cosine series. Illustrating this on a graph,



This is a somewhat intriguing result. Although we would have expected that each Fourier cosine and sine series would give different series, never did we anticipate that they would converge to different points in the same domain of the same function. Yet, this result does have it uses. When we need a half-range expansion to represent a real world phenomenon, finding a series, which converges to the point we *want*, does mean choosing the appropriate use of cosine or sine terms.