Fourier Analysis

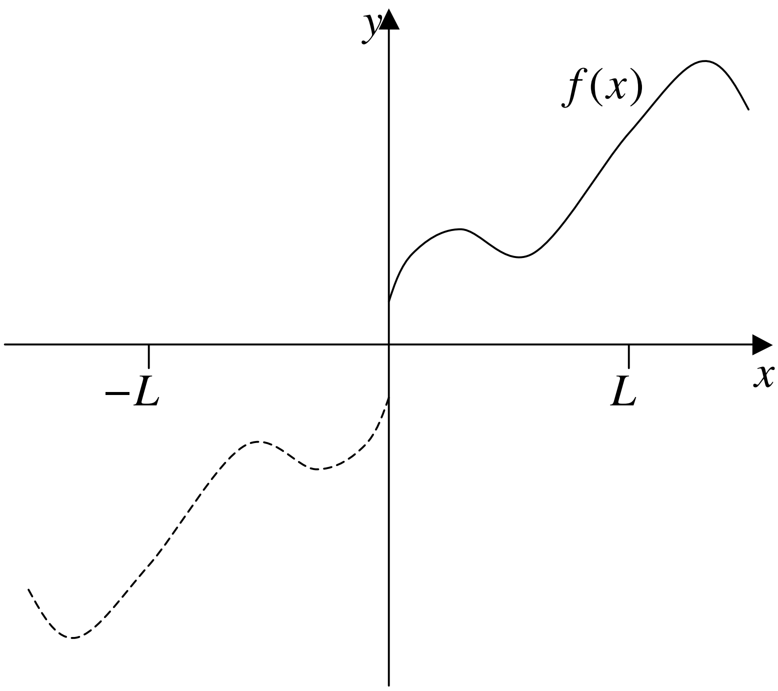
Fourier Sine Series

Continuing with half-range expansions, we now entertain the possibility of writing a Fourier sine series of f on $[0,L]$ The method is similar to that just used.

We extend f to an odd function w defined on $[-L,L]$ by letting

$$w(x)=\begin{cases} f(x) & \text{for } 0\le x\le L \\ -f(-x) & \text{for } -L\le x\le 0 \end{cases}$$

Below is a typical graph of w.



Since w is an odd function, its Fourier series on $[-L,L]$ contains only sine terms and is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

in which

$$b_n=\frac{2}{L}\int_0^L w(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Since *g*(x)=*f*(x) for $0 \le x \le L$,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Moreover on the interval [0,*L*]. we may take the Fourier series of *w* to be a Fourier sine series of *f*, as stated in the following definition.

If *f* is integrable on [0,*L*], the *Fourier sine series* of *f* on [0,*L*], is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

As usual, we form a convergence test for the Fourier sine series.

Let *f* be piecewise continuous on [0,*L*].
1. At any point *x* in (0,*L*) at which *f* has left and right derivatives, the Fourier sine series on [0,*L*] converges to

$$\tfrac{1}{2}\left[f(x+) + f(x-)\right],$$

the average of the left and right limits of *f* at *x*. In particular, if *f* is also continuous at *x,* the series converges to *f*(*x*).
2. At both zero and *L*, the sine series of *f* converges to zero.

Conclusion (2) is immediate upon letting *x*=0 and *x*=*L* in the sine series; all of the terms vanish because $\sin(0) = \sin(n\pi) = 0$ for any integer *n*.