Fourier Analysis

Fourier Series of x^2 [MAY NOT = f(x)]

Using the properties of odd and even functions, we shall now look at some short cuts we can take when finding the Fourier Series of an even function, in this case $f(x) = x^2$.

The objective is to find the Fourier series of $f(x) = x^2$ on [-3, 3]. Since x^2 is even on [-3, 3], we already know $b_n = 0$. That saves us a lot of working. We can now move on the find the other coefficients.

$$a_0 = \frac{1}{2 \cdot 3}\int_{-3}^{3} x^2 dx = \frac{1}{3}\int_{0}^{3} x^2 dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = 3$$

and for *n* = 1, 2, 3, …,

$$a_n = \frac{1}{3}\int_{-3}^{3} x^2 \cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_{0}^{3} x^2 \frac{d}{dx}\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\left[x^2 \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]_0^3 - \int_0^3 2x\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)dx\right\}$$

At this stage of the calculation, notice that the evaluated integrated on the left is equal to 0 by the identity $\sin(n\pi) = 0$. Be sure to make use for these ‘hidden’ identities. Now we are left with evaluating the integral on the right.

$$a_n = -\frac{4}{n\pi}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right)dx = -\frac{4}{n\pi}\int_0^3 x\frac{d}{dx}\frac{-3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= -\frac{4}{n\pi}\left(\left[x\frac{-3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_0^3 + \frac{3}{n\pi}\int_0^3 \cos\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{36}{n^2\pi^2}(-1)^n$$

since $\cos(n\pi) = (-1)^n$. The Fourier series of $f(x) = x^2$ on [-3, 3] is

$$3 + \sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{3}\right)$$

Remember way back in the earlier lessons on Fourier analysis, I emphasize the point to not be too quick in equating the function with the Fourier series. Well, let us take a look at that aspect using this function as an example.

Let *x* = 3. The function gives us $f(3) = 9$. From the Fourier series, we know that we will have the 3 on the first term. What about the summation term? By letting *x* = 3, we evaluate it as

$$\sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}(-1)^n \cos(n\pi) = \sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}(-1)^{2n} = \sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}$$

At our level, we have don't have a way of evaluating this summation and all we can *hope* is that it is equal to 6. How about we try another value of *x*.

This time, let *x* = 3/2. Clearly from the function, $f(3/2) = 9/4$. Again, we look at the summation term. This time, by substituting this value of *x*, the summation becomes,

$$\sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi}{2}\right)$$

Now things get a little more unclear. The value of $\cos(n\pi/2)$ can either be -1, 0 and 1. Moreover, the $36/n^2\pi^2$ term doesn't make things any easier for us. We have no idea at this point in confirming whether the Fourier series equals to 9/4.

Finally, let us look at another separate example. The Fourier series of $f(x) = x$ on [-3, 3] is given by

$$\sum_{n=1}^{\infty} \frac{6}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{3}\right)$$

We shall look at the endpoints of the Fourier series by substituting the value of x = 3. Our function gives us $f(3) = 3$. However, putting x = 3 into the Fourier series, we get 0 because $\sin(n\pi) = 0$. It is now obvious at the function and Fourier series does not agree at this point. This sets the stage for a convergence theorem, a test as you may it, that tells use at what domain does the Fourier series actually converges to the function itself.