

Fourier Analysis

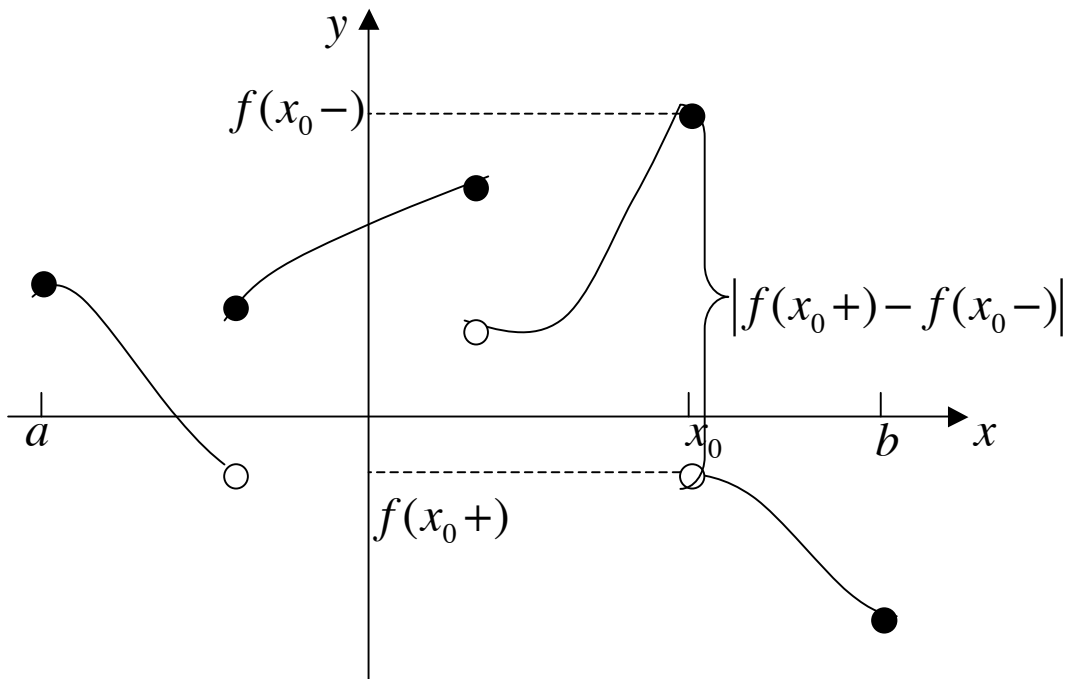
Recap on Piecewise Continuous functions

The previous lesson noted the motivational argument for the convergence theorem to tell us when the Fourier series converges to the function. Before we develop on this theorem, we need to equip ourselves with some definitions on piecewise continuous functions as we will be using them very soon.

Recall that a function $f(x)$ is *piecewise continuous* on $[a, b]$ if

1. $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ are finite.
2. $f(x)$ is continuous at all but possibly finitely many points of $[a, b]$.
3. At any point in (a, b) where $f(x)$ is discontinuous, the function has finite left and right limits.

The figure below illustrates the above points.



At points in (a, b) where the function is discontinuous, the graph has jump discontinuities. The magnitude of the jump at x_0 is the difference between the left and right limits there:

$$\left| \lim_{h \rightarrow 0^+} f(x_0 + h) - \lim_{h \rightarrow 0^-} f(x_0 + h) \right|$$

This is the distance between left and right “ends” of the graph at x_0 . We will denote left and right limits as follows:

$$f(x_0 -) = \lim_{h \rightarrow 0^+} f(x_0 - h) \text{ and } f(x_0 +) = \lim_{h \rightarrow 0^+} f(x_0 + h)$$

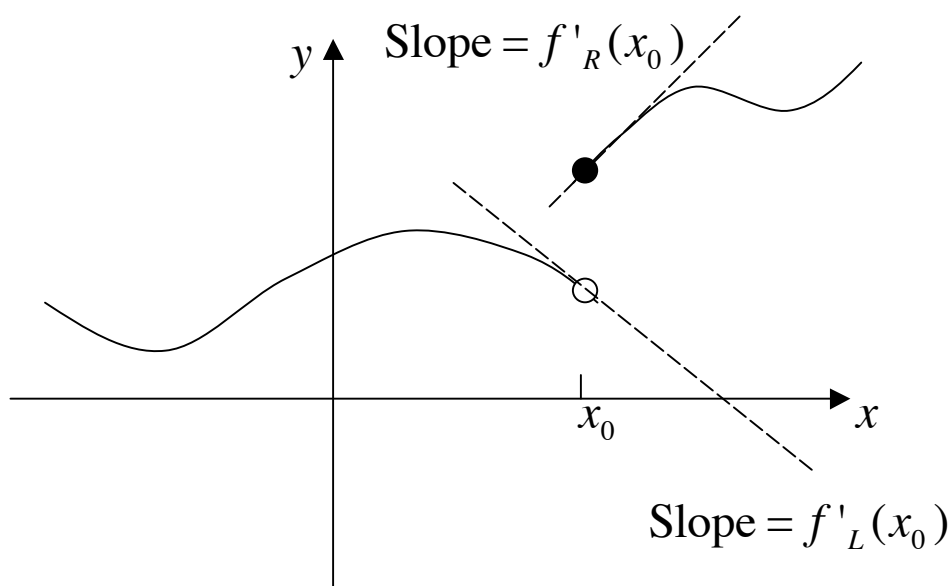
If $f(x)$ has a right limit at x_0 , the *right derivation* of $f(x)$ at x_0 is defined to be

$$f'_R(x_0) = \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0 +)}{h}$$

when this limit exists and is finite. We interpret $f'_R(x_0)$ as the slope of the tangent to the graph at x_0 if we look at only the portion of the graph to the right of x_0 . The *left derivative* of $f(x)$ at x_0 is

$$f'_L(x_0) = \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0 +)}{h}$$

when this limit exists and is finite. This number is the slope of the graph of $f(x)$ at x_0 , looking at only that portion to the left of x_0 . This is illustrated below.



We see that the left and right derivations of a function are given by the gradient of the respective dotted lines. At a jump discontinuity, the left and right derivatives may exist, while the derivative does not exist. Think of it as the graph having a tangent “from the left” and “from the right” at x_0 while having no tangent line at x_0 .

Some notes of evaluating these limits. Currently, we seem to be taking the limiting values when we approach integers from left or the right. Notice carefully that when evaluating the left and right limits, we let h tend towards 0 from the positive side alone but when finding the left and right derivatives, we let h tend towards 0 from the positive side for the left derivative and from the negative side for the right derivatives. While it may seem trivial now, after all h tends towards 0 anyways, I believe we should maintain the mathematical rigor throughout the lessons, even if the importance of which may seem irrelevant at present.