AMC

Functions 1980 AHSME #14

Suppose that the function f, for $x \neq -3/2$, is defined by

$$f(x) = \frac{cx}{2x+3}$$

and that $f(f(x)) = x$ for all real numbers in its domain. What is the value of c?

Solution

This question gets a little tricky because at first sight, it seems that we are solving for two variables, *x* and *c*, with only one equation. In a way, it seems that way, but the truth is that the variable we want to solve is *c* as posed by the question. What I suspect would happen is that we will find an equation to solve and somehow 'eliminate' *x* from that equation. That equation will come from the given $f(f(x)) = x$.

Here is where my tip with dealing with functions comes in handy. It is vital that you substitute the 'function of a function' properly. My tip is to rewrite the function for a clearer picture.

$$f(\_) = \frac{c\_}{2\_+3}$$

Basically, we will put whatever is in the blank in the '( )' into the blanks on the right hand side. So, it should be clear that

$$f(f(x)) = f\left(\frac{cx}{2x+3}\right) = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}$$

$$= \frac{cx^2}{2cx+6x+9}$$

Given that this is also equal to x, we have the equation

$$2cx^2 + 6x^2 + 9x = cx^2$$

or

$$0 = (2c + 6)x^2 + (9 - c^2)x$$

This is where our anticipated problem occurs, how are we going to solve for both *c* and *x* from this one equation. Well, with some algebraic manipulation, we can get through this problem.

The equal sign plays an important role here. If we can somehow pick a value of *c* such that we can turn both the coefficients of x^2 and x to zero, we would have made the left hand side equal to the right hand side, regardless the value of *x*. Let's try. We set $2c + 6 = 0$ giving us $c = -3$ and $9 - c^2 = 0$ giving us $c = -3$ or $c = 3$. Thus, we know that $c = -3$ as it satisfies the equation.

So our final answer is $c = -3$. I shall comment a little in saying that this works because we are at 'liberty' to pick a value for *c* which leads to us setting the appropriate equations.

Now, here comes an intriguing part. Another approach to the problem is to eliminate *x* by picking a real value for *x* and then solving for *c*. Maybe you would like to start with $x = 0$ but this gives us $f(f(0)) = 0$ regardless the value of *c*. So how about $x = 1$? It turns out to be

$$1 = f(f(1)) = f\left(\frac{c}{5}\right) = \frac{c^2/5}{2c/5 + 3} = \frac{c^2}{2c + 15}$$

and

$$0 = c^2 - 2c - 15 = (c - 5)(c + 3)$$

which leads to $c = -3$ and $c = 5$. Surprising result? I wonder how did the $c = 5$ come as it wouldn't satisfy $0 = (2c + 6)x^2 + (9 - c^2)x$.

This is that rare occasion where letting *x* take a certain value doesn't work out. A capable student could probably devise a method to eliminate the $c = 5$ as a solution but it is probably not worth the effort. Thus, I suggest solving the problem using the general method.