AMC

Functions 1991 AHSME #21

For all real numbers x, except $x=0$ and $x=1$, the function f is defined by

$$f\left(\frac{x}{x-1}\right)=\frac{1}{x}$$

Suppose $0<\theta<\pi/2$. What is $f\left((\sec\theta)^2\right)$?

Solution

Unlike the more usual functions that take the form $f(x)=$ an expression in terms of *x*, this function we are dealing with takes the form $f(x/x-1)$. While a little intimidating, you should be able to solve it quite easily with a solid foundation on definitions.

Our goal is to find $f\left((\sec\theta)^2\right)$, so if we could somehow pick a certain expression and substitute it into $x/x-1$ and then get $(\sec\theta)^2$, we are on our way because we simply substitute that same expression into $1/x$. However, the problem lies in finding that expression.

To efficiently solve this problem, we use a technique called ‘the switching of variable’, commonly used in other areas of algebra. We introduce a new variable to simplify the functional relationship to the point where we can immediately put $(\sec\theta)^2$ into the function much like how we put a value of *x* into the function $f(x)$.

Let $y=x/(x-1)$ and solve for *x* in terms of *y*. When $x\neq 0$ and $x\neq 1$ we have

$$y(x-1)=x$$

$$yx-y=x$$

$$yx-x=y$$

Solving for *x* in terms of *y* gives

$$x(y-1)=y$$

$$x=\frac{y}{y-1}$$

with the imposed condition that $y \neq 1$. We can now substitute both equations of *x* and *y* into the function.

$$f(y)=f\left(\frac{x}{x-1}\right)=\frac{1}{x}=\frac{1}{y/(y-1)}=\frac{y-1}{y}=1-\frac{1}{y}$$

We have simplified the functional relationship so now we can immediately let $y=(\sec\theta)^2$, for $0<\theta<\frac{\pi}{2}$, giving us

$$f\left((\sec\theta)^2\right)=1-\frac{1}{(\sec\theta)^2}=1-(\cos\theta)^2=(\sin\theta)^2$$

which is the answer.