AMC

Functions 1993 AHSME #26

The real-value function f is defined by

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}$$

What is the maximum value of $f(x)$?

Solution

In this question, $f(x)$ is given by two quadratic terms under square roots which is a killer to simplify and so answer to this question requires a different approach. One way is to consider completing the square for the individual quadratic terms.

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) \\ &= -(x^2 - 8x + 16) + 16 \\ &= 16 - (x-4)^2 \end{aligned}$$

and

$$\begin{aligned} 14x - x^2 - 48 &= -(x^2 - 14x + 48) \\ &= -(x^2 - 14x + 49) + 1 \\ &= 1 - (x-7)^2 \end{aligned}$$

For x to be in the domain of $f(x)$, we must have both

$$0 \le 16 - (x-4)^2 \text{, which implies}$$
$$|x-4| \le 4 \quad \text{or} \quad 0 \le x \le 8$$

and similarly

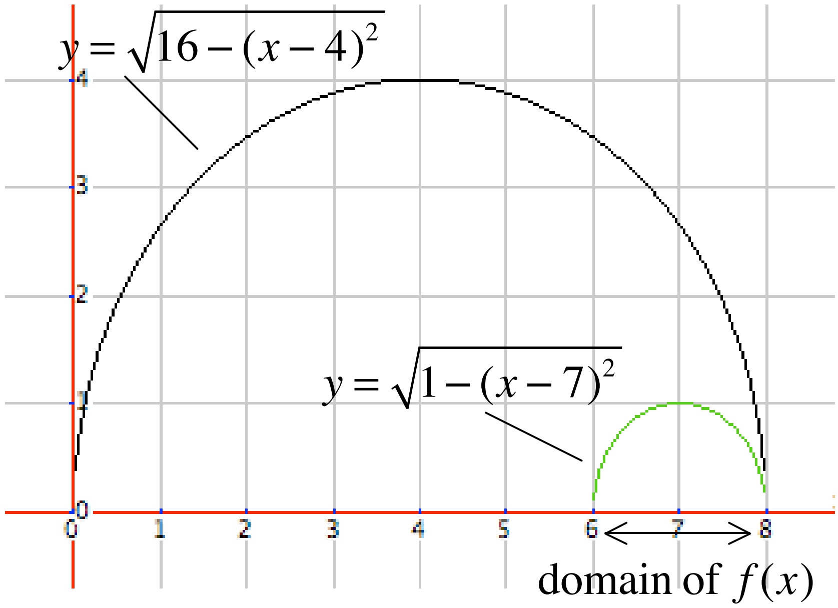
$$0 \le 1 - (x-7)^2 \text{, which implies}$$
$$|x-7| \le 1 \quad \text{or} \quad 6 \le x \le 8$$

We consider the values of x in the interval [6,8] and for these values we have

$$f(x)=\sqrt{16-(x-4)^2}-\sqrt{1-(x-7)^2}$$

At this juncture, we switch to a graphical analysis as our algebraic methods may prove insufficient.

Graphing out $y=\sqrt{16-(x-4)^2}$ and $y=\sqrt{1-(x-7)^2}$ separately.



As shown in the figure, the graph of $y=\sqrt{16-(x-4)^2}$ is the semi-circle in the first quadrant with the center at (4,0) and radius 4. The graph of $y=\sqrt{1-(x-7)^2}$ is the semi-circle in the first quadrant with the centre (7,0) and radius 1.

Since our function is valid in the domain [6,8], we only need to look at that portion of the graph which greatly simplifies our analysis. In this way, it is clear that the value in [6,8] that maximizes $f(x)$ is $x=6$ since this value maximizes $y=\sqrt{16-(x-4)^2}$ and also minimizes $y=\sqrt{1-(x-7)^2}$. hence, the maximum value of $f(x)$ is

$$f(6)=\sqrt{16-(6-4)^2}-\sqrt{1-(6-7)^2}=\sqrt{12}-\sqrt{0}=2\sqrt{3}$$

This is not such an easy question. Students of calculus may be tempted to use differentiation to find the maximum value. While this is perfectly valid, it may be tedious and prone to error for students who are not proficient in the technique. It suffices to say that no problem on the AMC has a calculus solution that is easier than some non-calculus solution.