AMC

Functions 1996 AHSME #12

The function f is defined for positive integers n by:

$$f(n)=\begin{cases} n+3, & \text{if } n \text{ is odd,} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Suppose k is an odd integer and that $f(f(f(k)))=27$. What is the sum of the digits of k?

Solution

First and foremost do not get intimidated by the 'function of a function of a function'. This question is easier than it looks. All it requires is a careful implementation of the function f. What you have to pay close attention to is the function is defined differently based on whether n is odd or even.

Since k is an odd integer, we have $f(k)=k+3$, which is an even integer. So

$$f(f(k))=f(k+3)=\frac{k+3}{2}$$

But now there are two possibilities to consider. We have no way of knowing if this last value is even or odd. We consider the cases separately.

If $f(f(k))$ is even, then

$$27=f(f(f(k)))=\frac{(k+3)/2}{2}=\frac{k+3}{4}$$
$$k=4\cdot 27-3=105$$

If $f(f(k))$ is odd, then

$$27=f(f(f(k)))=\frac{k+3}{2}+3$$
$$k=(2\cdot(27-3))-3=45$$

So which of these values is correct. We will simply work recursively putting these values into the original function $f(x)$ and see whether it leads to 27.

For $x = 105$,

$$f(f(f(105))) = f(f(108)) = f(54) = 27$$

which is correct.

For $x = 45$,

$$f(f(f(45))) = f(f(48)) = f(24) = 12$$

which is incorrect.

So the correct value is $k = 105$, whose sum of the digits is 6. This questions the irregularities that occur when applying the function continuously giving us a result which does not necessarily satisfy the initial condition.