Vector Integral Calculus
Tangent Plane

Using the previous theorems, we shall specify a method in finding the tangent plane to a level surface.

Remember that although we can't sketch the function $\varphi(x,y,z)$, we can sketch a level surface of that function given by $\varphi(x,y,z)=c$.

We will let $\varphi(x,y,z)=z-\sqrt{x^2+y^2}$ as our function to study. The level surface we will use is $\varphi(x,y,z)=0$ which gives us the cone

$$z=\sqrt{x^2+y^2}$$

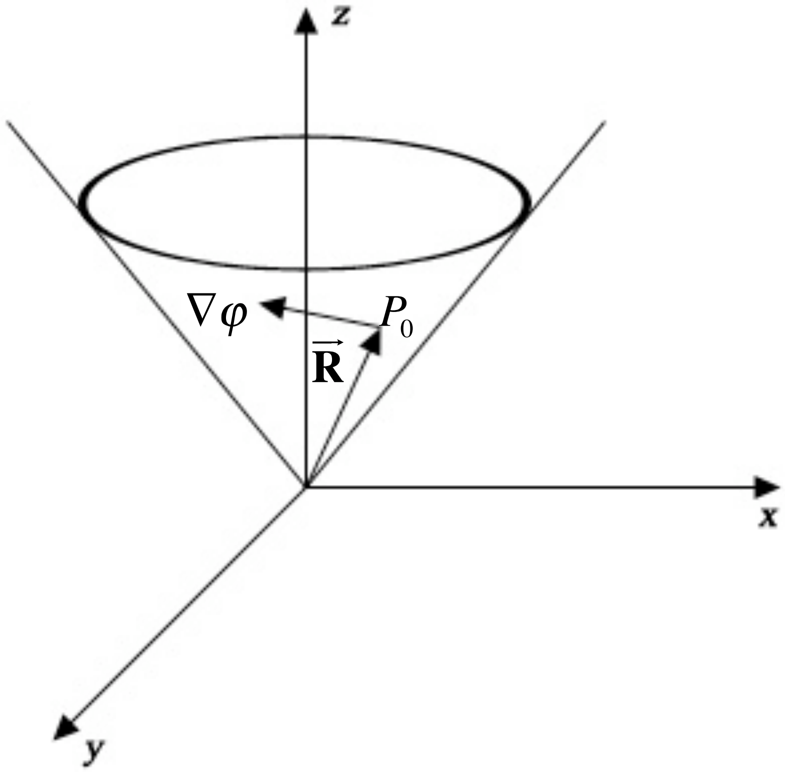
We then use the del operator on φ.

$$\nabla\varphi=\frac{-x}{\sqrt{x^2+y^2}}\mathbf{i}+\frac{-y}{\sqrt{x^2+y^2}}\mathbf{j}+\mathbf{k}$$

$$=-\frac{x}{z}\mathbf{i}-\frac{y}{z}\mathbf{j}+\mathbf{k}$$

where $\sqrt{x^2+y^2}\neq 0$

Other than the origin, the gradient $\nabla\varphi$ can be drawn as an arrow point into the cone and perpendicular to the side of the cone at any one point.
In other words, $\nabla\varphi(P)$ is perpendicular to the position vector $\vec{\mathbf{R}}(x,y,z)$ at (x,y,z) as shown below.



Explicitly, we can also show that these two vectors are indeed perpendicular by calculating their dot product.

$$\vec{\mathbf{R}} \bullet \nabla\varphi = \left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \bullet \left(-\frac{x}{z}\mathbf{i} - \frac{y}{z}\mathbf{j} + \mathbf{k}\right)$$

$$= -\frac{1}{z}x^2 - \frac{1}{z}y^2 + z = \frac{-x^2 - y^2 + z^2}{z} = 0$$

since on the surface, $z^2 = x^2 + y^2$.

We can use the gradient vector to find the tangent plane to the level surface $\varphi(x, y, z) = c$ at any point P_0 where the gradient is not 0 and is defined. This can be done by simple vector algebra.

We know that (x, y, z) is on the tangent plane and vector $(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is a vector parallel to the tangent plane and hence is perpendicular to $\nabla\varphi(P_0)$ and so

$$\nabla\varphi(P_0) \bullet \left[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}\right] = 0$$

After multiplying, this gives us the equation

$$\left.\frac{\partial\varphi}{\partial x}\right|_{P_0}(x-x_0)+\left.\frac{\partial\varphi}{\partial y}\right|_{P_0}(y-y_0)+\left.\frac{\partial\varphi}{\partial z}\right|_{P_0}(z-z_0)=0$$

which turns out to be our equation for the tangent plane at P_0.