

Vector Integral Calculus

Directional Derivative and the Del Operator

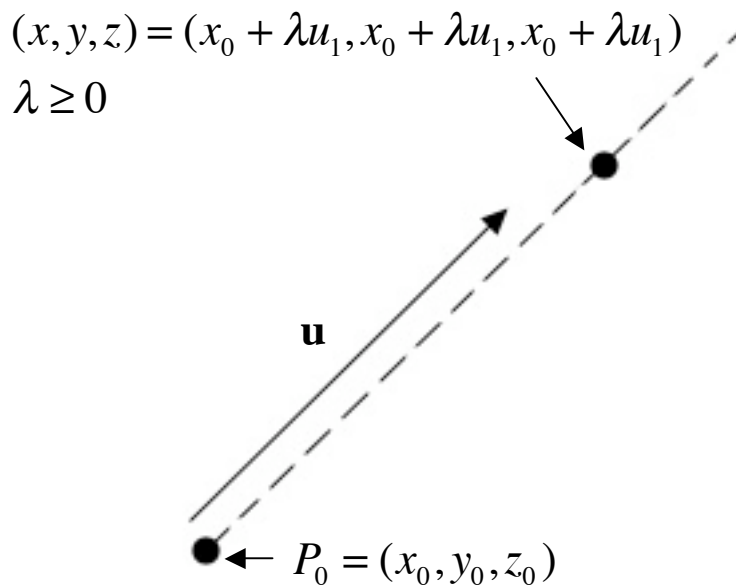
We'll follow up the previous lesson by deriving an equation which connects the directional derivative and the del operator.

While the definition of the directional derivative may suggest that it is hard to calculate, it turns out that it can be expressed quite elegantly.

Previously, we concluded with an expression of φ as

$$\varphi(x, y, z) = \varphi(x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3)$$

The graphical illustration of (x, y, z) can be seen from the previous diagram or replicated below.



We are concern with the rate of change of φ as we travel along unit vector \mathbf{u} . So, using process of differentiation, we find the derivative of $\varphi(x, y, z) = \varphi(x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3)$ with respect to λ , evaluated at $\lambda = 0$. Thus

$$D_u \varphi(P_0) = \frac{d}{dt} [\varphi(x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3)] \Big|_{\lambda=0}$$

Now, using the partial differentiation version of chain rule, we have

$$\begin{aligned}
 & \frac{d}{dt} [\varphi(x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3)] \Big|_{\lambda=0} \\
 &= \frac{\partial \varphi}{\partial x} \frac{\partial x}{\partial \lambda} + \frac{\partial \varphi}{\partial y} \frac{\partial y}{\partial \lambda} + \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial \lambda} \\
 &= \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial \lambda} (x_0 + \lambda u_1) + \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial \lambda} (y_0 + \lambda u_2) + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \lambda} (z_0 + \lambda u_3) \\
 &= \frac{\partial \varphi}{\partial x} u_1 + \frac{\partial \varphi}{\partial y} u_2 + \frac{\partial \varphi}{\partial z} u_3
 \end{aligned}$$

For the rate of change of φ at point P_0 , we are left to evaluate these partial derivatives at $\lambda = 0$ or $(x, y, z) = (x_0, y_0, z_0)$ then

$$D_u \varphi(P_0) = \frac{\partial \varphi}{\partial x} \Big|_{\lambda=0} u_1 + \frac{\partial \varphi}{\partial y} \Big|_{\lambda=0} u_2 + \frac{\partial \varphi}{\partial z} \Big|_{\lambda=0} u_3$$

Can you spot the grad φ hidden somewhere there. First recall that

$$\nabla \varphi = \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k},$$

a vector and that

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$$

also a vector. The dot product of these two vectors gives us the above expression, namely

$$D_u \varphi(P_0) = \nabla \varphi(P_0) \cdot \mathbf{u}$$

And that my friends, is how the directional derivative is related to the gradient of φ through the del operator. Lovely expression isn't it?

Remember that the vector \mathbf{u} needs to be a unit vector. This can be easily solved by letting $\mathbf{u} = \mathbf{v} / \|\mathbf{v}\|$ if traveling to P_0 with a vector \mathbf{v} whose magnitude is not unit.