Vector Integral Calculus

General and particular solution to Lines of Force

We will go through a short example of finding the lines of force of a particular vector field. I wish to make clear the two types of solutions – the general solution and the particular solution. Another important aspect here are the steps used to solve the differential equation.

Our given vector field is $\vec{\mathbf{F}}(x,y,z) = z^3\mathbf{i} + xy^2\mathbf{j} + 3\mathbf{k}$.

Let $\vec{\mathbf{F}}(x,y,z) = z^3\mathbf{i} + xy^2\mathbf{j} + 3\mathbf{k}$. We wish to find the lines of force of this vector field. From our previous lesson, we know that the lines of force will satsify the differential equation

$$\frac{dx}{z^3} = \frac{dy}{xy^2} = \frac{dz}{3}$$

Our task is thereforce to solve this equation. The most important step is to pick which set of equations you want to deal with first. The obivous choice is

$$\frac{dx}{z^3} = \frac{dz}{3}$$

reason being that using the dy differential will give us a mismatch of terms, we can't integrate either *x* w.r.t *z* or *z* w.r.t *y*, knowing that *x, y,* and *z* could each be expressed in terms of *x, y,* or *z*.

Separating the variables and integrating

$$\int 3\,dx = \int z^3 dz$$

$$3x = \frac{1}{4}z^4 + c_b - c_a$$

$$x = \frac{1}{12}z^4 + \frac{c_b - c_a}{3}$$

$$x = \frac{1}{12}z^4 + c_0$$

My way of dealing with the arbituary constants is to group them on one side and that equate them to a final arbituary constant to simplify the algebra.

We shall now deal with the equation

$$\frac{dy}{xy^2} = \frac{dz}{3}$$

as we have expressed *x* in terms of *z* so integrating w.r.t to *z* is now possible.

On separating the varaibles and integrating,

$$\int \frac{3}{y^2}\, dy = \int \frac{1}{12} z^4 + c_0 dz$$

Please note that the c_0 is important because when we integrate it w.r.t *z*, it will become the coefficient of a *z* term.

$$\frac{-3}{y} + c_a = \frac{1}{60} z^5 + c_0 z + c_b$$

$$\frac{-3}{y} = \frac{z^5 + 60c_0 z + 60c_c}{60}$$

$$y = \frac{-180}{z^5 + 60c_0 z + 60c_1}$$

where c_1 is the final arbituary constant we will use.

While you can choose to express the solutions in either *x*, *y* and *z*, we have chosen to express them in *z* simply because *z* is raise to certain powers, in this case 4 and 5. This is much neater as expressing the solutions in another variable may lead to taking fraction powers. It just goes to show it pays to express the solution in the ideal variable.

Thus our general solution to the differential equestion is

$$x = \frac{1}{12}z^4 + c_0$$
$$y = \frac{-180}{z^5 + 60c_0 z + 60c_1}$$
$$z = z$$

For our paricular solution, we are concern with the line of force that passes through the point $P(2,1,6)$. We substitute the values $x = 2,\ y = 1,\ z = 6$ into the equations to find c_0 and c_1 which in this case is

$$c_0 = -106$$
$$c_1 = 30204$$

Remember earlier in this chapter we said that there are an infinite amount of lines of force following this vector field. Well, this is exactly what the general solution tells us. To find that particular line of force, we substitute the *x,y,z* coordinates of that point.