Vector Integral Calculus

The Gradient Vector Field

Let $\varphi(x, y, z)$ be a real-value function of three variables. In the context of vectors, this function is called a *scalar field*. The gradient of φ, denoted by grad φ or $\nabla\varphi$, is a vector field extracted from φ according to the operation,

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}$$

where each of these partial derivatives are defined. The notation $\nabla\varphi$ is read as “del phi” and ∇ is called the *del operator*. While the geometric meaning of this operator will be elaborated later, just think of it as turning a scalar field to a vector field.

It also works on functions of two variable where φ is a function of *x* and *y*, in which case

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j}$$

Ensure that you are taking partial derivatives which basically means holding the other variable fixed and differentiating accordingly to the w.r.t term, i.e., ∂x means holding *y* fixed and differentiating w.r.t to *x*.

$\nabla\varphi(P_0)$ means the gradient of φ evaluated at P_0.

Let us look at a simple example. Suppose

$$\varphi = \frac{xy}{z^2 - 1}$$

then

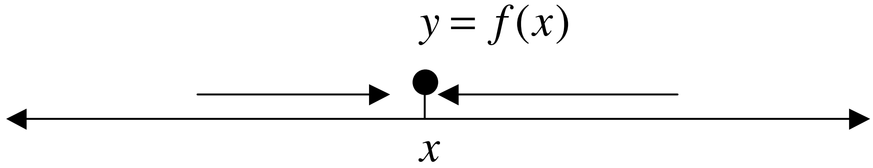
$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}$$

$$= \frac{y}{z^2-1}\mathbf{i} + \frac{x}{z^2-1}\mathbf{j} + \frac{-2xyz}{(z^2-1)^2}\mathbf{k}$$

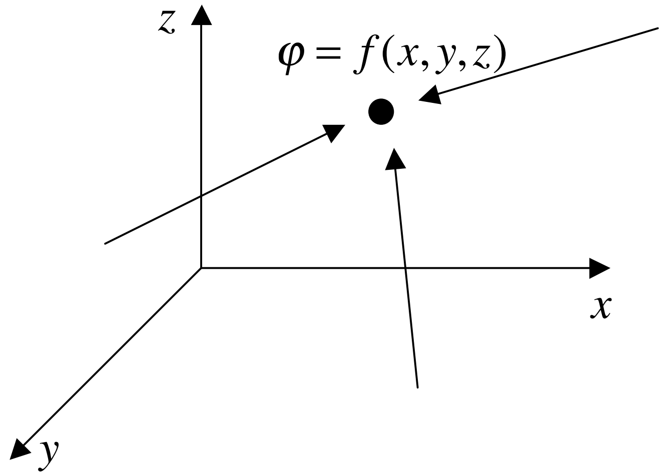
I will now define a new term which you guys may be initially confused at. Don't worry, you will be more familiar with it as we proceed in our learning.

The gradient of φ is related to the *directional derivative of* φ. What is this directional derivative? Suppose we are given a point P_0 in the direction of $\mathbf{u}$ specifying a direction from P_0. The directional derivative of φ at P_0 in the direction of $\mathbf{u}$ is the rate of change of φ with respect to (x, y, z) as (x, y, z) varies in the direction of $\mathbf{u}$ from P_0. We denote this directional derivative as $D_u\varphi(P_0)$. The reason for the underlining is because the definition needs to have all three terms, φ, P_0 and $\mathbf{u}$. In order to under this a bit better, I will compare it with the derivative as we know in 1 variable calculus.

In single variable calculus, the derivative of y measures the rate of change of y was we vary x. And in the number line, x can be varied in 2 ways, approaching it from the positive side or from the negative side as illustrated.



Now, let's us move into 3-dimensional space and think of the vector analogy. In vectors, a point P_0 in space is specificed by (x, y, z) and the function we apply is φ. We are to measure the rate of change of φ but this time, in 3-dimensional space, we approach and pass through P_0 using a vector, a vector which can originate from a variety of ways in the space and NOT limited to a number line like before. We designate this vector, unit vector $\mathbf{u}$, as illustrated.



And this is exactly what the directional derivative means: rate of change of φ with respect to (x, y, z) as (x, y, z) varies in the direction of $\mathbf{u}$ from P_0.

Just think of it in this way. When I fly my F-22 through a point in space, I will experience a rate of change of turbulence depending on which direction I choose to travel through that point. So, if I pull up and travel through that point vertically up, the rate of change of turbulence will be different than if I just fly horizontally through it. It all depends on the vector I choose, or the unit vector $\mathbf{u}$.

$$D_u \varphi(P_0)$$

directional derivative in the direction of this unit vector

The rate of change of the function at this point

the function we are measure the rate of change of

While I understand that this lesson is about the gradient vector field, we shall soon see how this links with directional derivative. Right now, just get the meaning of the directional derivative.