Vector Integral Calculus

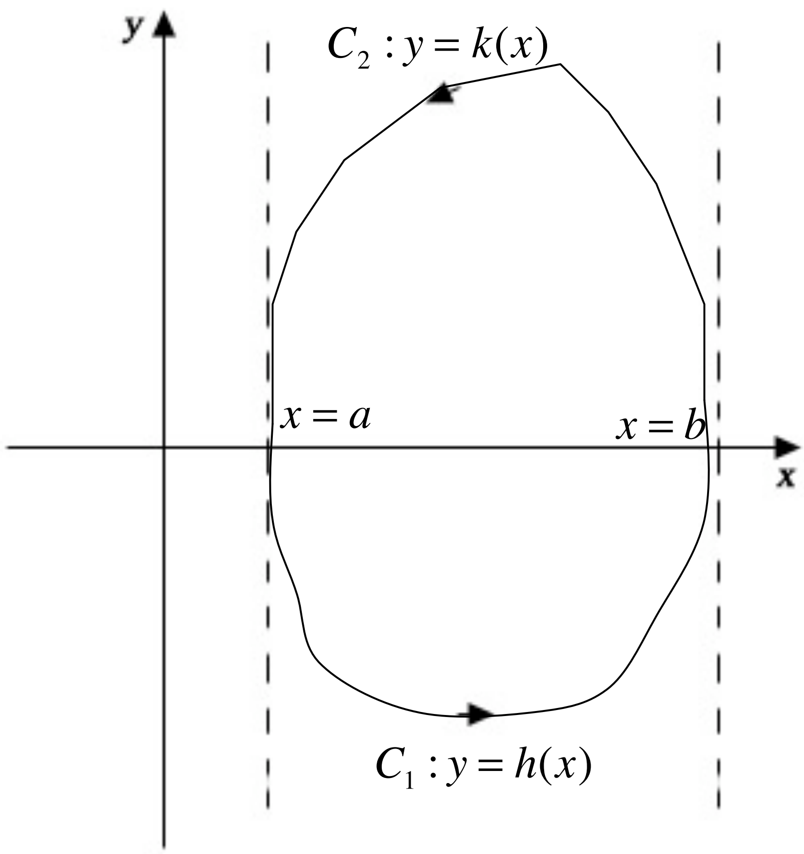
Proof of Green's Theorem

We shall conclude this discussion with a proof of a special case of Green's theorem. There are many extensions to Green's theorem but we will limit our proof to a certain type of curves.

Assume that *D* can be described in two ways. First, *D* has an upper portion described by the graph $y = k(x)$ and a lower portion described by the graph $y = h(x)$. Then *D* consist of all points (x, y) with

$$h(x) \le y \le k(x), \quad a \le x \le b$$

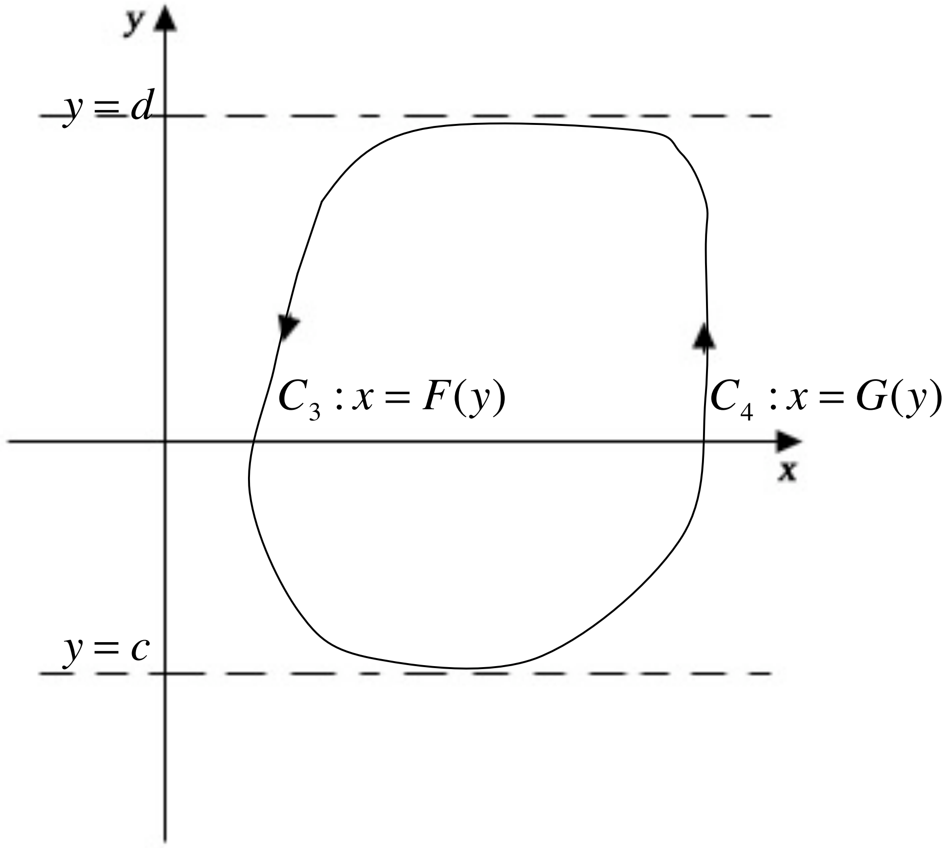
as shown below.



Second, *D* can also be described in another way with a left portion of the graph $x = F(y)$ and a right portion of the graph $x = G(y)$. Then *D* consist of all (x, y) with

$$F(y) \le x \le G(y), \quad c \le y \le d$$

as shown below.



We use these descriptions to show explicitly that

$$\oint_C f(x,y)\,dx = -\iint_D \frac{\partial f}{\partial y}\,dA$$

$$\text{and } \oint_C g(x,y)\,dx = \iint_D \frac{\partial g}{\partial x}\,dA$$

We get Green's theorem by adding these equations. Let us first look at the first equation.

From our first description of *D*, we can think of *C* comprising of $C_1 : y = h(x)$ with *x* varying from *a* to *b* and $C_2 : y = k(x)$ with *x* varying from *b* to *a*, maintaining the counter clockwise orientation. Note here that direction, and in turn limits of the integral, is important.

The strategy here is to evaluate the left hand and right hand side of the equation separately, using properties of the line integral for piecewise smooth curves, and show that they lead to a similar equation.

$$\begin{aligned}\oint_C f(x,y)\,dx &= \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx \\ &= \int_a^b f(x,h(x))dx + \int_b^a f(x,k(x))dx \\ &= \int_a^b f(x,h(x))dx - \int_a^b f(x,k(x))dx \\ &= \int_a^b -\left[f(x,k(x)) - f(x,h(x))\right]dx\end{aligned}$$

$$\begin{aligned}\iint_D \frac{\partial f}{\partial y}\,dA &= \int_a^b \int_{h(x)}^{k(x)} \frac{\partial f}{\partial y}\,dy\,dx \\ &= \int_a^b \left[f(x,y)\right]_{h(x)}^{k(x)}\,dx \\ &= \int_a^b \left[f(x,k(x)) - f(x,h(x))\right]dx\end{aligned}$$

Now that both the left hand and right hand side of the equation simplifies to the same expression, less the minus sign, we conclude that

$$\oint_C f(x,y)\,dx = -\iint_D \frac{\partial f}{\partial y}\,dA$$

Using the other description of *D* and a similar argument we can arrive to the equation

$$\oint_C g(x,y)\,dx = \iint_D \frac{\partial g}{\partial x}\,dA$$

Finally, adding these two equations yields Green's Theorem.

Here are some of my comments. I emphasize again that students new to this theorem should *not* attempt to find a geometrical link between the close loop integral and the double integral. Instead, one should only see the theorem solely for the purpose of calculating a tedious close loop integral by transforming it to the double integral.

The proof underlines one important aspect with line integrals and that is direction plays an important role. When we say positively orientated, we place importance on the direction the curve is traveling. Only by considering the direction could we change the limits of the integral and get the intended result.