Vector Integral Calculus

Green's Theorem

When evaluating a certain line integral, we are occasionally faced with the difficulty in the integration, as the equation of the curve may be complicated and thus tedious to evaluate.

In such circumstances, we can use Green's theorem to help us with the calculations.

Green's theorem links a line integral around a closed curve with a double integral over the part of the plane enclosed by the curve. The geometric link may not be easily seen but what matters more is its practical use.

A curve *C* in the plane given by the position vector

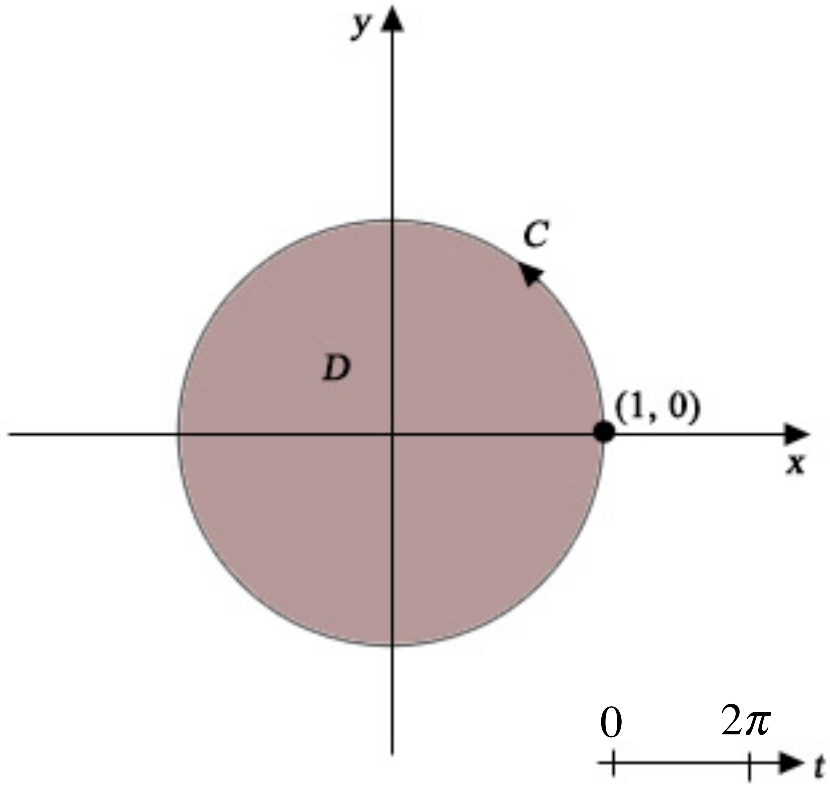
$$\vec{\mathbf{R}}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ a \le t < b.$$

Recall that *C* is closed if its initial point $(x(a), y(a))$ and terminal point $(x(b), y(b))$ are the same.

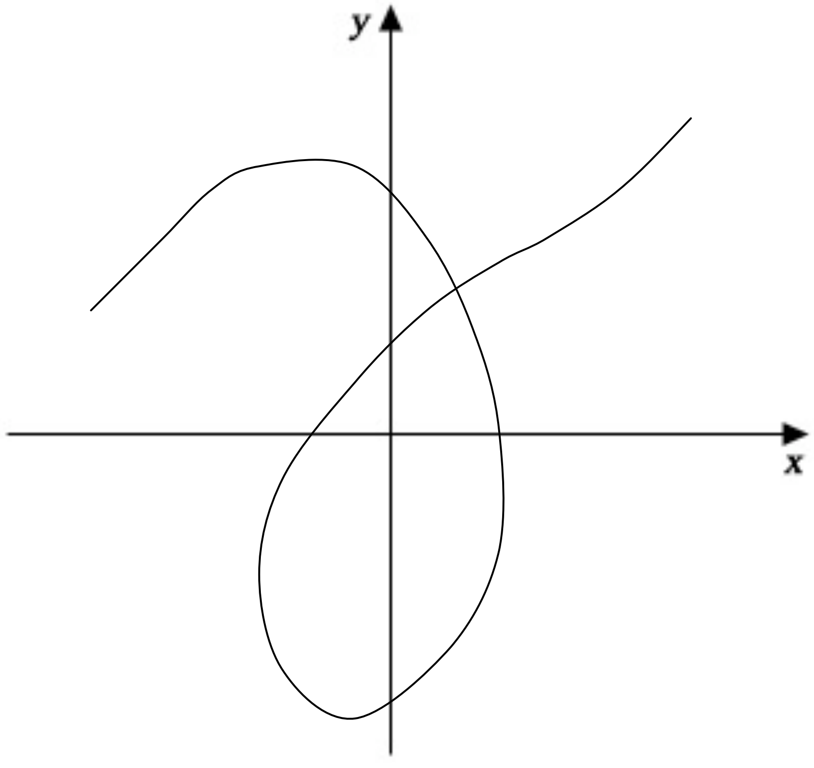
Before we proceed, it is vital that we introduce the concept of orientation of a curve. We say a closed curve *C* is *positively orientated* if $(x(t), y(t))$ moves around *C* counter-clockwise as *t* increase from *a* to *b.* If that's the case, then the set *D* enclosed by *C* is swept out on the person's left side as he walks around *C* in the positive sense. For illustration, let's look at the curve

$$\vec{\mathbf{R}}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ 0 \le t < 2\pi$$

The set *D* is given by the shaded area as shown below.

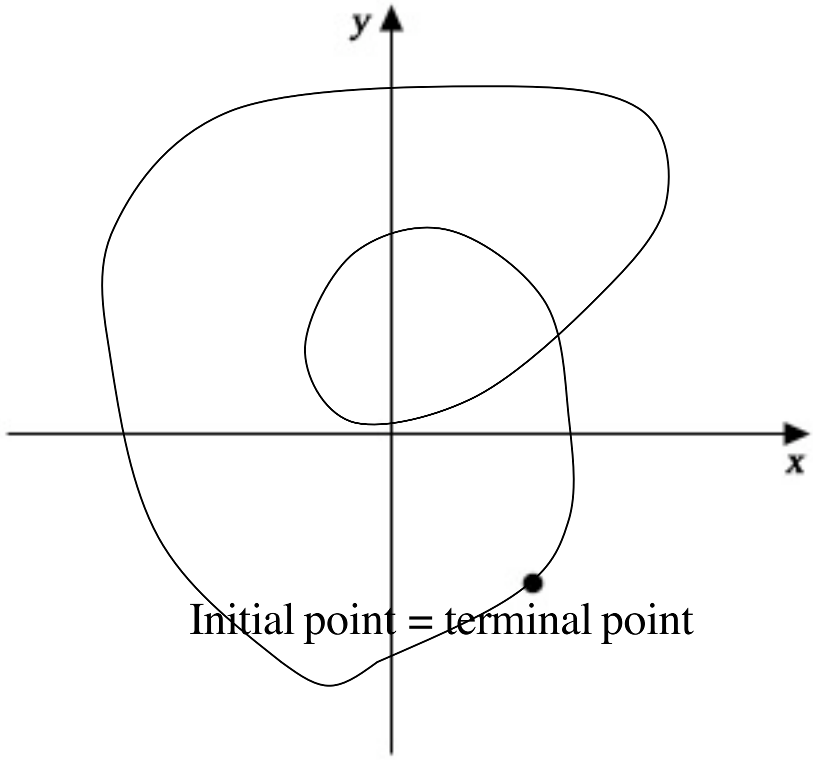


We call a nonclosed curve *C simple* if $\vec{\mathbf{R}}(t_1) \neq \vec{\mathbf{R}}(t_2)$ if $t_1 \neq t_2$ which means that a curve is simple if it does not cross itself. An example is shown below,



If *C* is closed, $\vec{\mathbf{R}}(a) = \vec{\mathbf{R}}(b)$, so *C* cannot be simple because $a \neq b$. However, we will still call *C* simple if the initial and terminal points are the only points which coincide for different values of the parameter, such as the curve of a circle which was shown at the beginning of the lesson.

A nonsimple closed curve is shown below, paying attention that the curve crosses itself at a point other than the initial and terminal points.

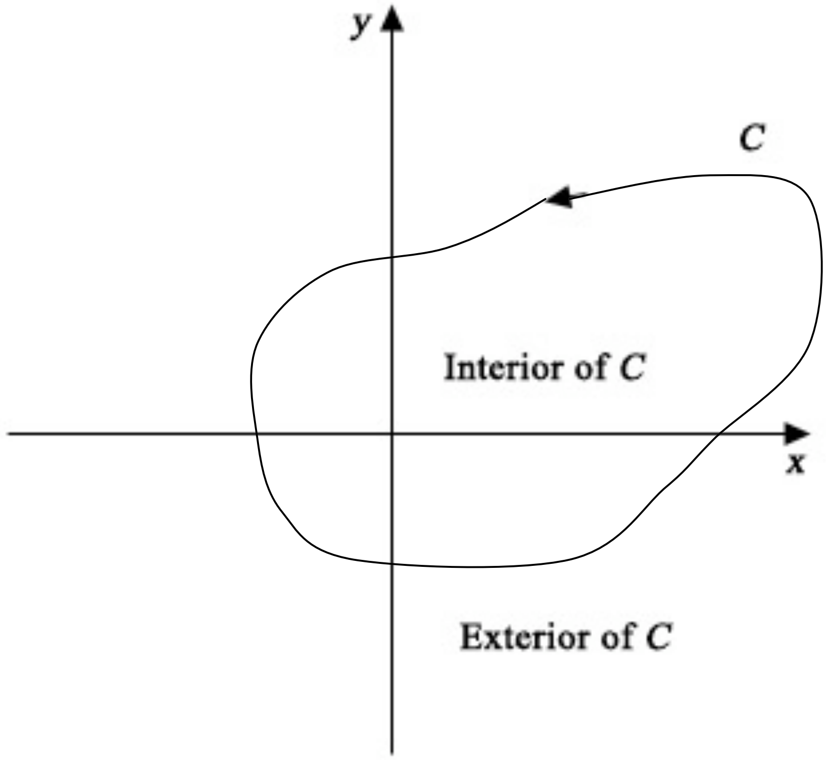


Just some quick change in notation: it's common to denote

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} \text{ as } \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$$

when *C* is a closed curve. The circle in the integral sign on the right is to simply remind the reader that the curve is closed loop. It has no bearing on how the integration is carried out.

The last term we need to be familiar with is the interior and exterior of a curve. A piecewise-smooth simple closed curve *C* in the plane divides the plane into two sets with *C* being the common boundary. One set contains points far from the origin called the *exterior of C.* The other set contains points inside of *C* called the *interior of C*.



Note that C itself does not belong to either of these sets but itself forms the common boundary between them.

With all these terms defined, we can now talk about Green's Theorem. We restrict our discussion in a plane.

Let C be a simple closed positively orientated piecewise-smooth curve in the plane. Let D consist of all points on C and in the interior of C. Suppose that $\vec{\mathbf{F}}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ is a continuous vector function whose components have continuous first partial derivatives through D. Then Green's theorem tells us that

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$$

or written as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$$

The theorem gives us an alternative way of evaluating line integrals.