Vector Integral Calculus

Uses of Green's Theorem

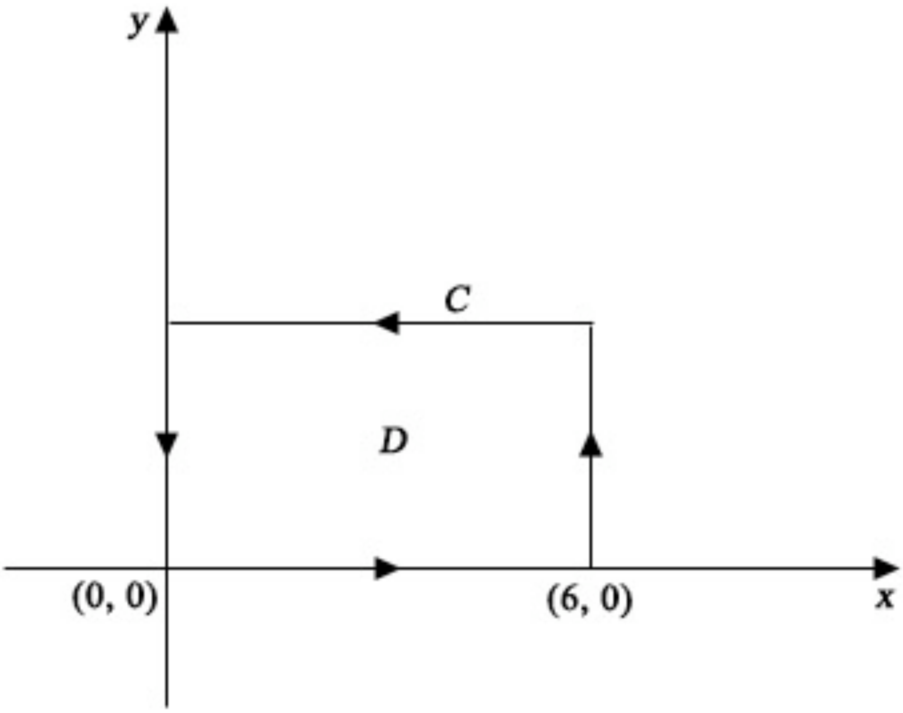
The main purpose of Green's Theorem is that it helps us find line integrals which will otherwise be difficult to integrate, in particularly piecewise-smooth curves where they are defined using multiple position vectors.

We shall look at one such example.

A particle moves in a counter clockwise direction around a rectangle having the vertices (0, 0), (6, 0), (0, 4), and (6, 4) under the influence of the vector field

$$\vec{\mathbf{F}} = x^2\mathbf{i} + 2xy\mathbf{j}$$

Our objective is to find the work done by $\vec{\mathbf{F}}$ after one complete circuit.



The work done is given by the close loop line integral

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$$

Noticing that the curve is piecewise-smooth, if we were to find this integral directly, we need to express the position vector $\vec{\mathbf{R}}$ into 4 parts, namely position vectors describing the edges of the curve.

An easier method is to use Green's theorem. Pay close attention that all the conditions are satisfied: *C* is simply closed positively oriented and piecewise-smooth AND the vector field has continuous first partial derivatives throughout *D*. So by Green's theorem, we can write,

$$\text{work done } = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$$

$$= \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2) \right] dA = \iint_D 2y \, dA$$

$$= \int_0^6 dx \int_0^4 2y \, dy = 6y\Big|_0^4 = 6 \times 16 = 96$$

using the appropriate rules of evaluating double integrals.

We learn here that given certain conditions, close loop line integrals can easily be calculated using Green's theorem.