Vector Integral Calculus
Definition of the Line Integral

After uploading on you a whole bunch of terms describing curves, let's put all of that into good use and define the line integral.

Before I proceed, I must say that you should separate any sort of notion of integration of real-value functions with the line integral. In terms of geometric meaning, they are both different. It is only the techniques of integrating the function which are the same.

Suppose we are given a smooth curve *C* and a vector field

$$\vec{\mathbf{F}}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$$

with component functions which are continuous over the graph of *C*. The *line integral of* $\vec{\mathbf{F}}$ *over C*, denoted by $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$, is defined to be

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = \int_a^b \vec{\mathbf{F}}(x(t), y(t), z(t)) \bullet \vec{\mathbf{R}}'(t)dt$$

Woah! You might be intimidated by the huge amount of terms in that single equation. Fret not, we shall go through the together.

First and foremost, let's clear up the algebra part of the $d\vec{\mathbf{R}}$. We know that $\vec{\mathbf{R}}$ is written in terms of *t*, so we can differentiate w.r.t *t* giving,

$$\frac{d\vec{\mathbf{R}}}{dt} = \vec{\mathbf{R}}'(t) \quad \text{and on rearranging}$$
$$d\vec{\mathbf{R}} = \vec{\mathbf{R}}'(t)dt$$

Secondly, when taking the line integral we need a vector field, in this case it's $\vec{\mathbf{F}}$ AND a curve defined by $\vec{\mathbf{R}}$. This is the first departure away from normal integration where we simply integrate a function. This time we need two vector functions. *C* in the symbol $\int_C$ specifies the curve.

Thirdly, the integration will be carried out with respect to the parameter *t*. You might be saying, 'wait a minute, I thought the vector field is not based on this parameter.' Well, you are correct. The vector field is in terms of a point (x, y, z) in the space. But by substituting the given coordinate functions $x = x(t), y = y(t), z = z(t)$, we get the relevant points on the curve and applying them to the vector field.

Thus, the process of evaluating $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$ can be reduced to three steps:

1. Form the dot product $\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}'$
2. Replace *x*, *y*, and *z* in $\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}'$ with the coordinate functions $x(t), y(t)$ and $z(t)$ of *C*.
3. Integrate the resulting function of *t* from *a* to *b*.

Only the last step involves techniques from normal calculus integration.

Right now, we are simply concern with the definition and not the geometric meaning of the line integral, which has quite a large scope and one of which will be discussed in two lessons from now. Also, I would like to mention that I may have left out the vector arrow above the vectors in writing some of the line integrals partly due to economy in writing. I apologies for the lack of consistency. This causes no difference in the expression, meaning to say $\vec{\mathbf{F}}, \vec{\mathbf{R}}$ is the same as $\mathbf{F}, \mathbf{R}$.