Vector Integral Calculus
Line Integral Example

In order for us to get acquainted with the line integral, let us look at a simple example.

Lets evaluate $\int_C \mathbf{F} \bullet d\mathbf{R}$ where $\vec{\mathbf{F}}(x,y,z) = x\mathbf{i} - yz\mathbf{j} + e^z\mathbf{k}$ and *C* is specified by $\vec{\mathbf{R}}(t) = t^3\mathbf{i} - t\mathbf{j} + t\mathbf{k}$ and $0 \le t \le 1$.

We first need to check whether the position vector is smooth. We have $\vec{\mathbf{R}}'(t) = 3t^2\mathbf{i} - \mathbf{j} + \mathbf{k}$ which is continuous in the interval so *C* is smooth. Calculating the dot product gives us

$$\begin{aligned}\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}' &= x(3t^2) - (yz)(-1) + e^z(1) \\ &= 3xt^2 + yz + e^z\end{aligned}$$

At this point, the vector field is still defined as *x*, *y*, and *z*. We need to substitute the component functions of $\vec{\mathbf{R}}$ as we are only concern with points on the curve *C*.

On *C*, $x(t) = t^3$, $y(t) = -1$ and $z(t) = 1$. Substituting into $\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}'$ gives us

$$\begin{aligned}\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}' &= 3t^3t^2 + (-t)(t) + e^t \\ &= 3t^5 - t^2 + e^t\end{aligned}$$

We can now integrate this function w.r.t *t* in the interval from 0 to 1.

$$\begin{aligned}\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} &= \int_0^1 (3t^5 - t^2 + e^t)\,dt \\ &= \frac{1}{2}t^6 - \frac{1}{3}t^3 + e^t\Big|_0^1 = \frac{1}{2} - \frac{1}{3} + e^1 - e^0 \\ &= -\frac{5}{6} + e\end{aligned}$$

We sum up by saying again to evaluate the line integral, we find the dot product $\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}'$, substitute the component functions in terms of *t*, and integrate w.r.t to *t*.