

Vector Integral Calculus

Interpretation of the Line Integral

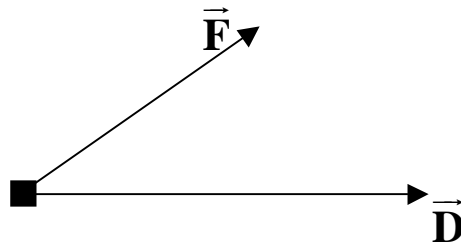
I believe it is time we have an interpretation to the line integral. So far, we have seen the conditions of the line integral and the technique we use to evaluate it, which is nothing more than finding the dot product, substituting the component functions for x , y , and z and integrating accordingly.

We now borrow some principles in physics for its interpretation.

In our study of vector fields in vector calculus, recall that the vector \vec{F} is a vector field that gives a vector at any point in three-space. In physics, we say that this force is acting on an object in the space. It doesn't require much intuition when I quote an example of say walking in water. Whatever point your body is in the water, there is a force that is acting on your body, the same force given by \vec{F} .

When we introduce the concept of a curve, we say that this is a force moving an object along C from its initial point $\vec{R}(a)$ to its terminal point $\vec{R}(b)$. Our main argument is that $\int_C \vec{F} \cdot d\vec{R}$ is the work done by this force.

Let's recall some physics. Suppose a constant force \vec{F} is acting on an object moving along vector \vec{D} as shown.



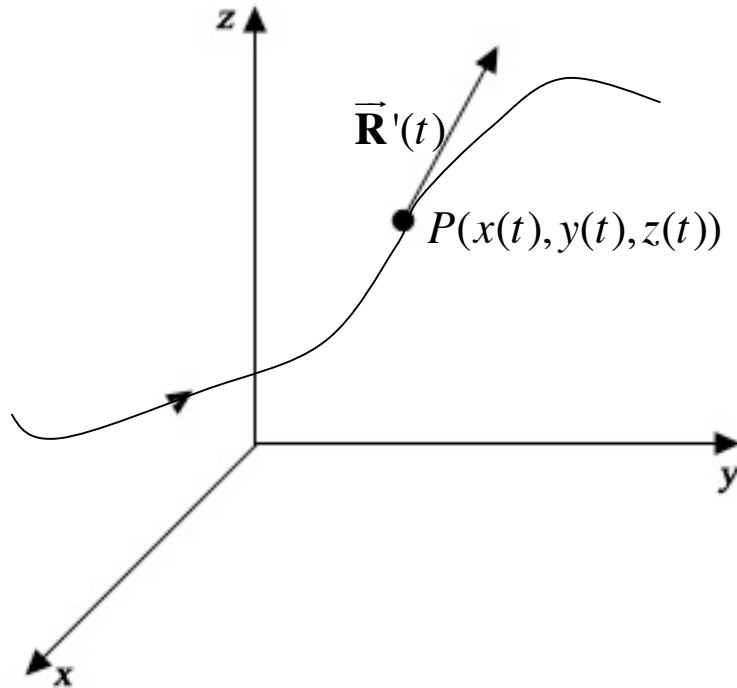
I drew the vector such that \vec{F} is *not* in the same direction as \vec{D} . Well, this can certainly be the case. Taking the walking in water example, I could be walking straight while the water is pushing me to the left. We are thus interested in the component of vector \vec{F} that is pushing the box along \vec{D} .

We know that the work done by this component of vector \vec{F} is given by

$$\vec{F} \cdot \vec{D}$$

Or in other words, the work done by \vec{F} to move the object along \vec{D} .

Going back to our curve C , we will derive an expression for the work done as follows. Choose a point $P(x(t), y(t), z(t))$ on the curve C .



The point will experience a force \vec{F} . To find the work done by this force at this instant, we need to know the direction of travel of point P at this same instant. Well, that isn't too hard to find because the direction of travel is simply given by the tangent vector or simply $\vec{R}'(t)$.

Therefore we sum $\vec{F} \cdot \vec{R}'$ over the entire curve by integrating giving us the conclusion that

$$\int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t) dt$$

is the total work done by the vector field \vec{F} . This is exactly the same as

$$\int_C \vec{F} \cdot d\vec{R}$$

which is the line integral.

From this discussion, we see that the work done by the vector field will be *different* if the object moves along another curve. This changes the $\vec{\mathbf{R}}$ in the given integral, in turn changing the work done.

This emphasizes again that the line integral needs both a vector *and* a position vector for it to make sense.