Vector Integral Calculus

Maximum and minimum rate of change

From our definition of the direction derivative, we can approach P_0 from a multitude of vectors, each giving a different rate of change of φ. In this section, we will find the vector $\mathbf{u}$ such that when we approach P_0 using this vector, we will get a maximum or minimum rate of change.

Suppose that $\varphi(x,y,z)$ is defined for all (x,y,z) in some sphere about a point P_0. As we approach point P_0 from different vectors, we find that φ is increasing in some directions, decreasing or remaing constant. To find the vector that gives us the maximum or minimum rate of change of φ, we simply use two simple theorems.

Let $\varphi(x,y,z)$ and its first partial derivatives be continuous in some sphere about P_0 and assume $\nabla\varphi(P_0) \neq \mathbf{O}$.

First theorem: The direction from P_0 in which φ has its maximum rate of change is in the direciton of $\nabla\varphi(P_0)$ and this maximum rate of change is $\|\nabla\varphi(P_0)\|$.

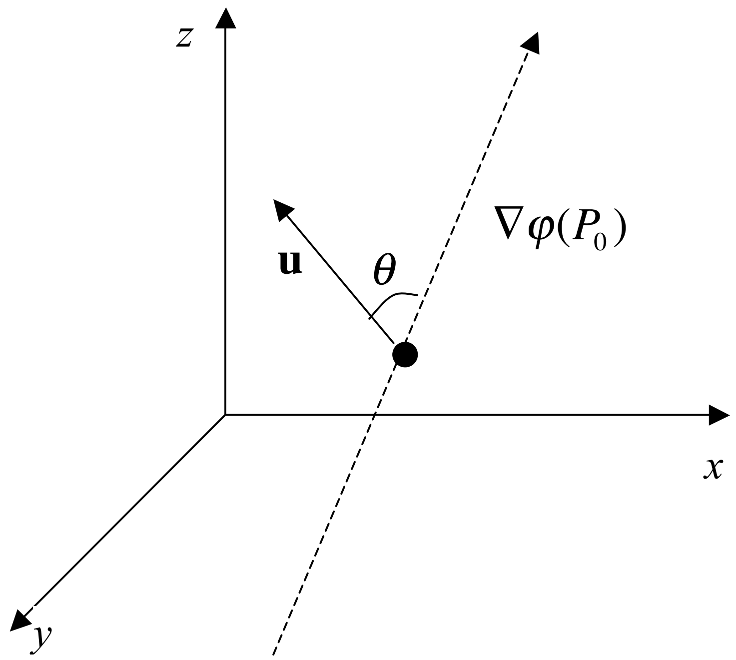
Second theorem: The direction from P_0 in which φ has its minimum rate of change is in the direciton of $-\nabla\varphi(P_0)$ and this minimum rate of cahnge is $-\|\nabla\varphi(P_0)\|$.

The two theorems are essentially opposites of each other and are not too hard to prove with the help of the cosine function. We shall show the proof here.

Let $\mathbf{u}$ be any unit vector drawn as an arrow from P_0. Then,

$$D_u\varphi(P_0) = \nabla\varphi(P_0) \bullet \mathbf{u}$$

Knowing that we can choose from different unit vectors, we want to choose $\mathbf{u}$ such that this directional derivative is as large as possible. We now define the angle θ to be the angle between $\mathbf{u}$ and $\nabla\varphi(P_0)$ as shown.



By using vector algebra, we write

$$\nabla\varphi(P_0)\bullet\mathbf{u} = \left\|\nabla\varphi(P_0)\right\|\left\|\mathbf{u}\right\|\cos(\theta) = \left\|\nabla\varphi(P_0)\right\|\cos(\theta)$$

since $\left\|\mathbf{u}\right\| = 1$. Here comes the reasoning. We maximize $D_u\varphi(P_0)$ by maximizing $\nabla\varphi(P_0)\bullet\mathbf{u}$ and in turn maximizing $\cos(\theta)$. This occurs when $\cos(\theta) = 1$ or $\theta = 0$ meaning to say that $\mathbf{u}$ is in the same direction as $\nabla\varphi(P_0)$. Moreover, this gives us $D_u\varphi(P_0) = \left\|\nabla\varphi(P_0)\right\|$ as our theorem says.

By the same argument, $\nabla\varphi(P_0)\bullet\mathbf{u}$ has its lowest value at $\cos(\theta) = -1$ or $\theta = \pi$. In this case, $\mathbf{u}$ is in the same direciton as $-\nabla\varphi(P_0)$ and $D_u\varphi(P_0) = -\left\|\nabla\varphi(P_0)\right\|$.

We use these two theorems to find normal vectors and tangent planes which are coming up next.