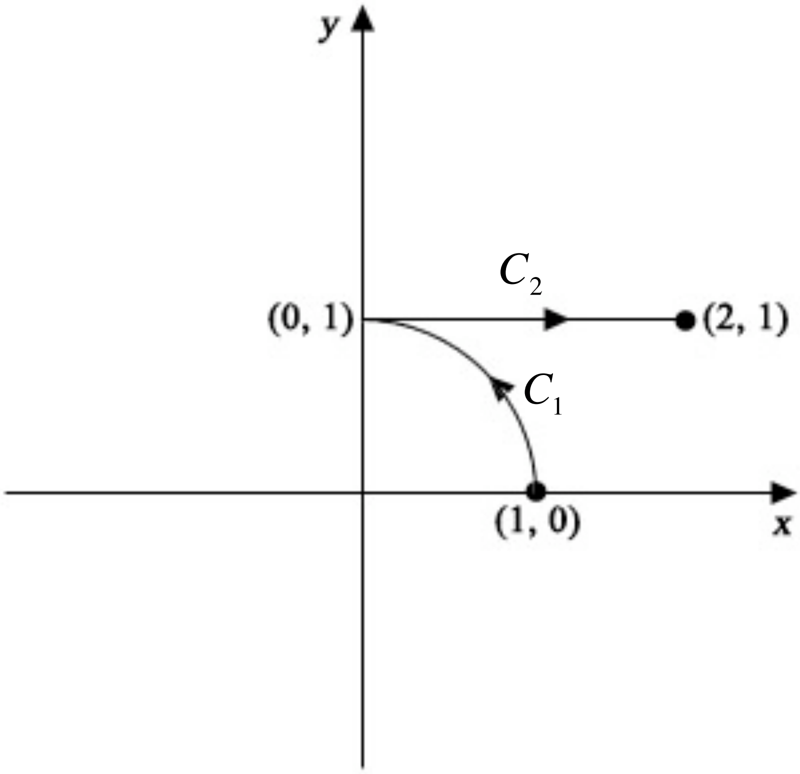
Vector Integral Calculus

Piecewise smooth curve example

To get familiar with calculating line integrals of piecewise smooth curves, we shall look at a simple example.

Let *C* be the curve traversing a quarter circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$ in the plane, then moving along the horizontal line from $(0,1)$ to $(2,1)$. We will work with a vector field $\vec{\mathbf{F}}(x,y,z) = 4x\mathbf{i}$. Our objective is thus to find $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$.

We limit our analysis to a plane because the vector field $\vec{\mathbf{F}}$ is independent of *z* and has zero **k**-component and *C* happens to be on a plane too.



In this situation, we need to split the piecewise smooth curve *C* into two separate curves C_1, C_2. We shall parameterize these individually as

$$C_1: \quad x = \cos(t), \quad y = \sin(t); \quad 0 \le t \le \tfrac{\pi}{2}$$
$$C_2: \quad x = t, \quad y = 1; \quad 0 \le t \le 2$$

bearing in mind that the interval in which the parameter varies is as equally important as the *x, y* functions.

We now evaluate $\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$ and $\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$ independently. For C_1,

the position vector is $\vec{\mathbf{R}} = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ and $\vec{\mathbf{R}}' = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$, and so

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}' = 4x\left[-\sin(t)\right] = -4\cos(t)\sin(t)$$

and

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = \int_0^{\frac{\pi}{2}} -4\sin(t)\cos(t)\,dt = -2\sin^2(t)\Big|_0^{\frac{\pi}{2}} = -2$$

We now do the same for the other curve C_2. The position vector is $\vec{\mathbf{R}} = t\mathbf{i} + \mathbf{j}$ and $\vec{\mathbf{R}}' = \mathbf{i}$ and so $\vec{\mathbf{F}} \bullet \vec{\mathbf{R}}' = 4x(1) = 4x = 4t$.

$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = \int_0^2 4t\,dt = 2t^2\Big|_0^2 = 8$$

We add these two results to give us the line integral for *C*, the piecewise smooth curve consisting of the two smooth curves C_1, C_2.

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}} = -2 + 8 = 6$$